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QUILLEN'S THEOREM ON BUILDINGS AND THE LOOPS ON A SYMMETRIC SPACE

by Stephen A. MITCHELL [1])

ABSTRACT

In the mid 70's Garland and Raghunathan, and (independently) Quillen discovered that ΩG (G a compact Lie group) is homotopy equivalent to an infinite dimensional flag variety, and that Bott's cell decomposition of ΩG can then be obtained as a Bruhat or Schubert cell decomposition. Quillen's method applies also to the loops on a compact symmetric space M, and involves identifying the path space of M with a certain Bruhat—Tits building. The details never appeared. In this paper we develop a theory of "topological" buildings and prove Quillen's theorem. We then show how one can rederive the Bott—Samelson theorems on ΩM, and the real and complex Bott periodicity theorems, from this point of view.

In the 1950's Bott, and Bott and Samelson, obtained a series of beautiful results on the topology of loop spaces of compact symmetric spaces: the Bott periodicity theorems [6], a cell structure (with various applications to homology) [4] [7], and a description of the Pontrjagin ring [5]. All of these theorems were proved using Morse theory. In the mid 70's another very different approach emerged in the work of Garland and Raghunathan [12] (who only consider the case of a compact Lie group) and (independently) Quillen [30]. The new point of view forms a part of the theory of loop groups: if G is a simply-connected compact Lie group, with complexification $G_{\mathbf{C}}$, the group $LG_{\mathbf{C}}$ of maps $S^1 \to G_{\mathbf{C}}$ can be regarded as an infinite dimensional complex algebraic group. The based loops ΩG then appear as a homogeneous space of $LG_{\mathbf{C}}$, analogous to a flag variety. If $M = G/K$ is a symmetric space, ΩM is a real form of ΩG. The cell structures of Bott and Samelson are obtained from a Bruhat decomposition of $LG_{\mathbf{C}}$, and their results can be derived from the combinatorics of the affine Weyl group. In addition, ΩM is the direct limit of finite dimensional "Schubert varieties", and recently this point of view has led to some new

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results on the homotopy type of $\Omega SU(n)$, $\Omega SU(n)/O(n)$, and $\Omega SU(2n)/Sp(n)$ ([9], [24], [26]).

There are two key ingredients. The first concerns the structure of the group of "algebraic loops" $L_{alg}G_{\mathbf{C}}$—i.e., the regular maps $\mathbf{C}^* \to G_{\mathbf{C}}$ (§ 3). Here the basic idea goes back to Iwahori and Matsumoto [16], following a suggestion of Bruhat. These authors show how to associate a Tits system (§ 2) to any Chevalley group over a local field, so that the Weyl group of the system is the affine Weyl group. Now if we write $\tilde{G}_{\mathbf{C}}$ for the group of points of the algebraic group $G_{\mathbf{C}}$ over $\mathbf{C}[z, z^{-1}]$, it is easy to see that $L_{alg}G_{\mathbf{C}} = \tilde{G}_{\mathbf{C}}$. Hence the results of [16] can be applied (at least after completing with respect to the ideal (z)) and we obtain a Tits system on $L_{alg}G_{\mathbf{C}}$. The group P of regular maps $\mathbf{C} \to G_{\mathbf{C}}$ is then a maximal parabolic subgroup, and the homogeneous space $\tilde{G}_{\mathbf{C}}/P$ (which is a direct limit of projective varieties) can be identified with $\Omega_{alg}G = \{f \in L_{alg}G_{\mathbf{C}}: f(S^1) \leqslant G \text{ and } f(1) = 1\}$ ("Iwasawa decomposition"). The axioms for a Tits system then yield a Bruhat or Schubert cell decomposition of $\tilde{G}_{\mathbf{C}}/P$, and hence a cell decomposition of $\Omega_{alg}G$. The cells are indexed by $\mathrm{Hom}(S^1, T)$, where T is a maximal torus. After some further technical work, this idea can be generalized to ΩM: if $M = G/K$, where $K = G^\sigma$ for some anti-complex involution σ on $G_{\mathbf{C}}$ preserving G, we can define an involution τ an $L_{alg}G_{\mathbf{C}}$ by $\tau f(z) = \sigma(f(\bar{z}))$. The fixed group is a real form of $\tilde{G}_{\mathbf{C}}$. Similarly, we replace $\Omega_{alg}G$ by $(\Omega_{alg}G)^\tau$—the space of $\mathbf{Z}/2$—equivariant loops—and the cell decomposition of $(\Omega_{alg}G)^\tau$ is obtained in an analogous way (§ 5). Of course to apply any of this to the original problem, we need the second ingredient: Let $\Omega_{alg}M = (\Omega_{alg}G)^\tau$, and note that ΩM can be identified with $(\Omega G)^\tau$.

THEOREM (Quillen). *The inclusion* $\Omega_{alg}M \to \Omega M$ *is a homotopy equivalence.*

In the case $M = G$ this theorem has several completely different proofs ([12], [29], [30]); it can also be deduced from Bott's work and the Bruhat decomposition [25]. The proof suggested by Quillen is particularly beautiful, and applies to all compact symmetric spaces M. The idea is the following, taking $M = G$ for simplicity: It is sufficient to produce a contractible space E on which $\Omega_{alg}G$ acts freely, with orbit space G. Quillen observes that a plausible candidate for E is already at hand. To any Tits system one can associate a certain simplicial complex (or space) $\mathscr{B}$—the *building*—and when the Weyl group of the system is infinite, as it is here, the building is contractible. In fact $\mathscr{B}$ is a certain quotient space of $L_{alg}G/T \times \Delta$, where Δ is a simplex of dimension equal to rank G. It follows that the $\Omega_{alg}G$ orbit space is a quotient of $G/T \times \Delta$. But it is a classical fact that $G = G/T \times \Delta/\sim$ (§ 1), and on inspection one sees that the two quotients are identical. In fact (this is also due to Quillen) $\mathscr{B}$ has a very concrete description: it is the space of paths in G of the form $f(e^{2\pi it}) \exp tX$, where $f \in \Omega_{alg}G$ and $X \in \mathfrak{g}$. The action of $\Omega_{alg}G$ on this space is obviously free, which completes the proof (§ 4).

The purpose of this paper is to give a detailed exposition of Quillen's idea, with reasonably complete proofs, and to show how one can derive the results of Bott and Samelson. (Along the way, we also give an axiomatic treatment of topological Tits systems.) The paper is organized as follows:

In § 1 we establish most of our notation concerning Lie groups, symmetric spaces, etc.; and collect some preliminary results. The most important point here is the classical description of M as a quotient of $K \times \Delta$, where Δ is the Cartan simplex. The reader will probably prefer to skim through this section first, and refer back to it later when necessary. The main references are [13], [22], and [33]; a short introduction to real forms, Satake diagrams, etc. can be found in [23].

In § 2 we discuss topological Tits systems (G, B, N, S) and their associated buildings. Although the axioms for a Tits system may seem obscure at first encounter, and lack the geometric appeal of Morse theory, it can not be denied that they are remarkably simple. The structure theory of such systems constitutes our main technical tool. However, in our context it is necessary to take into account the topology the system. We define topological Tits systems in a rather minimal way, and then state four additional axioms that will be satisfied by all the Tits systems considered in this paper. These axioms are fairly easy to verify in most cases, and suffice to establish various desirable properties: For example, that the Bruhat decomposition of a "flag space" G/P is a CW-decomposition, with the closure relations on the cells given by the Bruhat order on the Weyl group. Much of the treatment here is inspired by Steinberg ([32]) and Kac and Peterson ([17], [18], [19]). We then introduce the topological building $\mathfrak{B}_G$. It is a quotient space of $G/B \times \Delta$—in fact, it is precisely the homotopy colimit of the diagram of flag spaces $G/P_I (I \underset{\neq}{\subset} S)$. We show how to adopt the standard proofs of the Solomon-Tits theorem to the topological context. Thus $\mathfrak{B}_G$ is contractible if W is infinite and is a certain suspended quotient of G/B otherwise. As an example, we note that for the usual Tits system associated with a real form of a semisimple complex Lie group, the building can be identified with the "tangent cut locus" of the associated compact symmetric space.

In § 3 we briefly review some basic facts about algebraic loop groups. (See for example [1], [27] and [29] for details). The most important fact is that $LG_{\mathbf{C}}$ admits a suitable topological Tits system. The existence of the Tits system is proved more generally for Kac-Moody groups by Kac and Peterson [17], so we only sketch the proof.

In § 4 we prove Quillen's theorem on the building, in the case $M = G$. (We have separated this case from the general case in order to isolate the main idea, which is fairly simple.)

In § 5 we redo the results of § 3, 4 for a general M. Again, many of the more tedious technical results are only sketched. One key result is the existence of a suitable Bruhat decomposition of the real form $(L_{alg}G_{\mathbf{C}})^{\tau}$. Presumably this follows from the general theory of algebraic groups, but we have elected to give a direct proof that contains a result of some independent interest. The point is that the involution τ does not preserve the Iwahori subgroup $\tilde{B}$ (the "B" of the Tits system), so one can not simply apply τ to the $\tilde{B} - \tilde{B}$ double cosets in $\tilde{G}_{\mathbf{C}}$. However τ does preserve a certain parabolic $\tilde{Q}$ (canonically associated to the original involution σ), and hence preserves the $\tilde{Q} - \tilde{Q}$ double cosets. To analyze these, we show more generally that for any flag variety $G_{\mathbf{C}}/Q$ or $\tilde{G}_{\mathbf{C}}/\tilde{Q}$, the P-orbits (here P, Q are any parabolics) are holomorphic vector bundles over (finite dimensional) flag varieties of the Levi factor of P (which can be explicitly determined). This fact is certainly well known, but does not seem to appear in the literature. The details are banished to an appendix (§ 8). We also show in this section how to deduce various results from [7]: the cell structure on ΩM, the fact that these cells are all cycles mod 2 (or actual cycles, if M is of "splitting rank"), and the "somewhat mysterious" connection [7] between $H_*\Omega G$ and $H_*\Omega M$, when M is of maximal rank. (This connection becomes transparent in the present context.)

In § 6, we discuss six examples: $SU(2n)/Sp(n)$, $SU(n)/SO(n)$, $SO(2n)/U(n)$, $Sp(n)/U(n)$, S^n and $\mathbf{C}P^n$. Here, as elsewhere, we emphasize the way in which information can be obtained directly from the Satake and Dynkin diagrams.

In § 7, we reprove the real and complex periodicity theorems. In effect, we simply imitate Bott's original, beautiful proof, but with Morse theory replaced by topological Tits systems. The idea is that for certain commutator maps $K/H \xrightarrow{\varphi} \Omega G/K$, $\varphi(K/H)$ is a "Schubert subvariety", so the range of dimensions in which φ is an equivalence can be determined by merely counting cells. But as an added twist, we show that if one only considers the maps φ associated with the "miniscule roots" of M (these suffice for Bott periodicity), then this range of dimensions is not only determined by the root system (as Bott showed), but in fact can be read off directly, in a rather amusing way, from the Dynkin diagram. Thus the Bott periodicity theorems can be proved by inspecting the Dynkin diagrams of the classical symmetric spaces!

A traditional difficulty encountered by writers on this subject is the inordinate quantity of notation required: to the usual list of notations for root systems, Coxeter groups, complex Lie groups, etc., we must add still more notation for symmetric spaces, restricted root systems, loop groups, etc. Some further remarks: (1) we generally use a tilda for various "loop" analogues of classical objects, but this notation should be interpreted with care. For example, it $G_{\mathbf{C}}$ is a reductive complex algebraic group, $\tilde{G}_{\mathbf{C}}$ is the group of algebraic $G_{\mathbf{C}}$-valued loops; on the other hand, if B is a Borel subgroup of $\tilde{G}_{\mathbf{C}}$, its analogue is the Iwahori subgroup $\tilde{B}$—but $\tilde{B}$ is not a Borel subgroup, and is not the group of B-valued algebraic loops (see § 3). (2) in a similar vein, we generally use a subscript $\mathbf{R}$ to denote the analogue for a real form (given a fixed involution σ as above) of a complex object. For example, $G_{\mathbf{R}}$ is our real form of $G_{\mathbf{C}}$: $G_{\mathbf{R}} = (G_{\mathbf{C}})^{\sigma}$. On the other hand $B_{\mathbf{R}}$, the analogue of B, is usually called a "minimal parabolic". It is neither solvable nor connected in general, and does not equal B^{σ}, but nevertheless is the correct analogue of B (from the point of view of Tits systems). (3) Given a root system Φ (affine or ordinary), we frequently confuse, identify and otherwise comingle the following sets: (a) the simple roots (a system of positive roots having been fixed), (b) the simple reflections, (c) the nodes of the Dynkin diagram and (d) a set of integers $1, 2, \cdots l$ (*or* $0, 1, 2, \cdots l$) indexing all three of the above in a compatible way.

A final word on the origin of this paper: Quillen's work is unpublished, and, to the best of my knowledge, he never even circulated a manuscript. I first learned of the idea (of using the building) from a set of notes, kindly sent to me by Richard Kane, of a single lecture delivered by Quillen at MIT in July of 1975. Theorems 4.1, 4.2, 4.4 and 4.7 are stated there, and it is asserted that the methods and results carry over to symmetric spaces. The proofs of these theorems in the present paper are (for better or for worse) my own. The Bruhat and Iwasawa decompositions for algebraic loop groups (or at least their topological applications) are apparently due to Quillen and (independently) Garland and Raghunathan, although in their algebraic form these results go back to Iwahori and Matsumoto. The treatment here is largely based on work of Kac and Peterson [17]. Another approach is via the "Grassmanian model" representation of $\Omega_{alg}G$; this too is due to Quillen. We will not consider the Grassmanian model (or its obvious analogue for symmetric spaces, but see [9] for an example); there is a very thorough account of this approach in [29].

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§ 1. Notation and Preliminaries

Except in § 2, G will always denote a compact connected Lie group of rank l; usually we will assume also that G is simple and simply-connected. Fix once and for all a maximal torus T in G, and let N denote the normalizer $N_G T$. The Weyl group W is N/T. Lie algebras are denoted as usual by Gothic letter: $\mathfrak{g}$, $\mathfrak{t}$, etc. To each G we can associate a reductive complex algebraic group $G_{\mathbf{C}}$—the complexification of G—with Lie algebra $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes \mathbf{C}$. It contains G as a maximal compact subgroup, and as the fixed group of an anti-complex involution. In fact $G \to G_{\mathbf{C}}$ defines an equivalence of categories (compact Lie groups) $\leftrightarrow$ (reductive complex algebraic groups).

$G_{\mathbf{C}}$ has a Borel subgroup (maximal connected solvable subgroup) B, unique up to conjugacy, which we can assume contains the Cartan subgroup (maximal algebraic torus) $T_{\mathbf{C}}$. There is a split extension $U \to B \to T_{\mathbf{C}}$ where U is the unipotent radical of B. There is also an opposite Borel subgroup B^- such that $B \cap B^- = T_{\mathbf{C}}$; it fits into a similar split extension $U^- \to B^- \to T_{\mathbf{C}}$. On the Lie algebra level we have $\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \mathfrak{u} \oplus \mathfrak{u}^-$, with $\mathfrak{u} \oplus \mathfrak{u}^-$ being precisely the sum of the nontrivial eigenspaces for the adjoint action of $\mathfrak{t}_{\mathbf{C}}$ on $\mathfrak{g}_{\mathbf{C}}$. The corresponding eigenfunctions $\lambda : \mathfrak{t}_{\mathbf{C}} \to \mathbf{C}$ map $\mathfrak{t}$ into $i\mathbf{R}$; as is customary we replace each λ by $\alpha = \lambda/2\pi i$ to obtain a set Φ of nontrivial $\mathbf{R}$-valued linear functionals on $\mathfrak{t}$-the real roots. These form a (reduced, crystallographic) root system in $\mathfrak{t}^*$. The positive roots Φ^+ correspond to $\mathfrak{u}$, the negative roots Φ^- to $\mathfrak{u}^-$. A simple system of roots $\alpha_1, ..., \alpha_l$ (here we assume G is semisimple of rank l) is then uniquely determined as the set of positive roots which are not decomposable as sums of positive roots. If we assume G is simple, so that Φ is irreducible, there is a unique "highest root" α_0, which is characterized by the property that for every positive root α, $\alpha_0 + \alpha$ is not a root. The corresponding eigenspace in $\mathfrak{u}$ is precisely the center of $\mathfrak{u}$. And, speaking of eigenspaces, let X_α denote the eigenspace (or "root subalgebra") of $\mathfrak{g}_{\mathbf{C}}$ associated to $\alpha \in \Phi$. For each α, the subalgebra of $\mathfrak{g}_{\mathbf{C}}$ generated by X_α and $X_{-\alpha}$ is isomorphic to $sl(2, \mathbf{C})$. The corresponding subgroup, isomorphic to $SL_2\mathbf{C}$ or $PSL_2\mathbf{C}$, is $G_{\mathbf{C},\alpha}$. Choosing generators E_α for the X_α, we obtain a basis for $\mathfrak{g}_{\mathbf{C}}$, consisting of the $E_\alpha(\alpha \in \Phi)$ and $H_\alpha = [E_\alpha, E_{-\alpha}]$ $(\alpha \in \Phi^+)$.