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GLOBAL CONSTRUCTION OF THE NORMALIZATION OF STEIN SPACES

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INTRODUCTION

A fundamental tool in the theory of complex manifolds X is Riemann's Theorem on Removable Singularities of holomorphic functions which ensures that all functions holomorphic outside of a rare analytic subset of X and locally bounded on X can be extended to functions holomorphic on all of X. In other words, all weakly holomorphic functions on X are actually holomorphic. Although this theorem does not hold for arbitrary complex spaces, Oka [12] showed in 1951 that every complex space X can be modified to a complex space $\tilde{X}$ for which Riemann's continuation theorem is valid, the so-called normalization of X.

Stein spaces X are complex spaces which can be completely described by the algebra $\mathcal{O}(X)$ of global holomorphic functions. Since a complex space is Stein if and only if its normalization is Stein [11], it is natural to ask if the normalization $\tilde{X}$ of a Stein space X can be constructed just from the holomorphic functions on X. Phrased differently, the question is whether the algebra $\mathcal{O}(\tilde{X})$ of all holomorphic functions on $\tilde{X}$ or equivalently, the algebra $\tilde{\mathcal{O}}(X)$ of all weakly holomorphic functions on X, can be derived from the algebra $\mathcal{O}(X)$ of holomorphic functions on X.

The purpose of this paper is to demonstrate that this is possible when X is irreducible: $\tilde{\mathcal{O}}(X)$ is the topological closure of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$. An example given in § 1 shows that $\widetilde{\mathcal{O}(X)}$ is not in general topologically closed even if X is locally irreducible. $\tilde{\mathcal{O}}(X)$ can also be obtained by taking the intersection of the localizations $S_x^{-1}\,\widetilde{\mathcal{O}(X)}$ of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ with respect to $S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\}$ for every $x \in X$ (see § 3).

The proof relies on an analytic and an algebraic theorem, namely Rossi's theorem [13] generalizing the Remmert quotient and the integral closure theorem of Mori-Nagata [7].

An analytic consequence of the construction presented here is that the normalization $\tilde{X}$ of an irreducible Stein space X is $\widetilde{\mathcal{O}(X)}$-convex, $\widetilde{\mathcal{O}(X)}$-separable and has local coordinates by functions in $\widetilde{\mathcal{O}(X)}$. Some algebraic results are that $\mathcal{O}(\tilde{X})$ is completely normal and that the two algebras $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ are always locally equal, i.e. their localizations at all maximal ideals in $\mathcal{O}(X)$ are equal.

In this paper, a complex space refers to a reduced complex space with countable topology.

1. Example of a Stein space X with $\widetilde{\mathcal{O}(X)} \neq \mathcal{O}(\tilde{X})$

Let $(X, \mathcal{O})$ be a complex space with normalization $\pi: \tilde{X} \to X$. Since π is surjective, the map $\pi^*: \mathcal{O}(X) \to \mathcal{O}(\tilde{X})$, $f \mapsto f \circ \pi$, is injective and the holomorphic functions $\mathcal{O}(X)$ on X can be considered to be a subring of the holomorphic functions $\mathcal{O}(\tilde{X})$ on the normalization $\tilde{X}$ of X; this will be indicated by $\mathcal{O}(X) \subset \mathcal{O}(\tilde{X})$. If X is irreducible and Stein, then $\mathcal{O}(\tilde{X})$ contains the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ but does not always coincide with it, as will be shown in this section.

For an irreducible complex space $(X, \mathcal{O})$, the integral domain $\mathcal{O}(X)$ is said to be *normal*, if it is integrally closed in its field of fractions $Q(\mathcal{O}(X))$, i.e. $\widetilde{\mathcal{O}(X)} = \mathcal{O}(X)$. Recall that $Q(\mathcal{O}(X))$ is the field of meromorphic functions $M(X)$ on X when X is irreducible and Stein due to Theorem B [10, 53.1, 52.17], and that the algebras $M(X)$ and $M(\tilde{X})$ are isomorphic for every complex space X [8, p. 161].

The following characterization of normal irreducible Stein spaces X by their global function algebra $\mathcal{O}(X)$ is essentially contained in [2, § 1, p. 35].

Theorem 1. *An irreducible Stein space X is normal if and only if the integral domain $\mathcal{O}(X)$ is normal.*

An analysis of the proof shows that even when X is just irreducible and normal, $\mathcal{O}(X)$ is also normal. Theorem 1 implies

Corollary 1. *For an irreducible Stein space X with normalization $\tilde{X}$, the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ is contained in $\mathcal{O}(\tilde{X})$.*

The following example shows that there are functions $f \in \mathcal{O}(\tilde{X})$ which are not integral over $\mathcal{O}(X)$. In this example, $X := (\mathbf{C}, \mathcal{O}')$ is an irreducible and locally irreducible Stein space given by a substructure of the canonical complex plane $(\mathbf{C}, \mathcal{O})$, which is then the normalization $\tilde{X}$ of X. The substructure is defined by a "Strukturausdünnung" (see [10]) which results by replacing the stalks $\mathcal{O}_n$, $n \in \mathbf{N}$, with the stalks of generalized Neil parabolas becoming steeper as n increases. More precisely, let $(p_n)_{n \in \mathbf{N}}$ be a strictly increasing sequence of prime numbers. For every $n \in \mathbf{N}$,

$$X_n := \{(x, y) \in \mathbf{C}^2 : x^{p_n} = y^{p_n + 1}\}$$

is an irreducible, locally irreducible analytic subset of $\mathbf{C}^2$ with the origin as the only singularity and with normalization

$$\pi_n : \mathbf{C} \to X_n, \; t \mapsto (t^{p_n + 1}, t^{p_n}).$$

Let $f \in \mathcal{O}(\mathbf{C})$ be the identity and denote by $\mathcal{O}_{X_n}$ the canonical complex structure on X_n. The germ $f_0 \in \mathcal{O}_0$ of f at the origin is integral over $\mathcal{O}_{X_n, 0}$ with respect to a polynomial of degree p_n, and p_n is the minimal degree of all such polynomials.

Now define $X := (\mathbf{C}, \mathcal{O}')$ as a substructure of the canonical plane $(\mathbf{C}, \mathcal{O})$ with stalks

$$\mathcal{O}'_x \cong \begin{cases} \mathcal{O}_x, & x \notin \mathbf{N} \\ \mathcal{O}_{X_n, 0}, & x = n \in \mathbf{N} \end{cases}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}'_n & \rightarrow & \mathcal{O}_n \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{O}_{X_n, 0} & \xrightarrow{\pi_n^*} & \mathcal{O}_0, \end{array}$$

where $\mathcal{O}'_n \to \mathcal{O}_n$ is the map induced by the identity $(\mathbf{C}, \mathcal{O}) \to (\mathbf{C}, \mathcal{O}')$ and $\mathcal{O}_n \cong \mathcal{O}_0$ is determined by the translation $\mathbf{C} \to \mathbf{C}$, $z \mapsto z - n$.

The identity $f \in \mathcal{O}(\mathbf{C})$ is not integral over $\mathcal{O}'(\mathbf{C})$, because otherwise every polynomial of integral dependence would have degree at least p_n for all $n \in \mathbf{N}$.

In conclusion it should be mentioned that $\mathcal{O}(\tilde{X})$ is almost integral over $\mathcal{O}(X)$ [7, § 3] for every irreducible Stein space X, since X has a global universal denominator [10, E.73a].

2. Construction of $\mathcal{O}(\tilde{X})$ from $\mathcal{O}(X)$ for Stein spaces X

According to a theorem of Oka [12], the normalization sheaf $\tilde{\mathcal{O}}$ of weakly holomorphic functions on a complex space $(X, \mathcal{O})$ is coherent. Consequently, there is a canonical topology making $\tilde{\mathcal{O}}$ a Fréchet sheaf; the global weakly holomorphic functions $\tilde{\mathcal{O}}(X)$ will always carry this topology. Since the holomorphic functions $\mathcal{O}(\tilde{X})$ on the normalization $\tilde{X}$ of X are topologically isomorphic to $\tilde{\mathcal{O}}(X)$ [8, 8.3], the question posed in the introduction can now be answered.

Main theorem. *For an irreducible Stein space X, the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ is dense in $\tilde{\mathcal{O}}(X)$.*

Proof. Let $\pi: \tilde{X} \to X$ be the normalization of X and put $A := \widetilde{\mathcal{O}(X)}$. Since π is proper, $\tilde{X}$ is $\mathcal{O}(X)$-convex and therefore $\bar{A}$-convex. Note that Corollary 1 implies $A \subset \tilde{\mathcal{O}}(X)$ and that $\bar{A}$ is the closure of A with respect to the canonical topology in $\tilde{\mathcal{O}}(X)$.

Consider the equivalence relation R on $\tilde{X}$ defined by $\bar{A}$, i.e. $(x, y) \in R$ iff for every $f \in \bar{A}$, $f(x) = f(y)$. Rossi's theorem [13] ensures that the topological quotient $Y := \tilde{X}/R$ can be given the complex structure of a Stein space such that the projection $p: \tilde{X} \to Y$ is holomorphic and proper and the map $p^*: \mathcal{O}(Y) \to \mathcal{O}(\tilde{X})$, $f \mapsto f \circ p$, induces an isomorphism $\mathcal{O}(Y) \cong \bar{A}$.

It suffices to show that every $f \in \mathcal{O}(\tilde{X})$ can be factorized through a holomorphic function on Y, meaning that an $F \in \mathcal{O}(Y)$ exists with $F \circ p = f$. This will be accomplished by first factorizing $f \in \mathcal{O}(\tilde{X})$ through a continuous function F on Y and then proving that F is actually holomorphic. The existence of such a continuous factor F for f is equivalent to demonstrating that every $f \in \mathcal{O}(\tilde{X})$ is constant on the fibers of p. The validity of this geometric statement will be shown now using commutative algebra.

$\mathcal{O}(\tilde{X})$ is almost integral over $\mathcal{O}(X)$ (see § 1), and hence over the localization $S_x^{-1}A$ of A with respect to $S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\}$ for every $x \in X$. Moreover,

$$S_x^{-1}A = \widetilde{S_x^{-1}\mathcal{O}(X)}$$

holds [3, V, 1.5, Corollary 1]. The localization $S_x^{-1}\mathcal{O}(X) = \mathcal{O}(X)_{m(x)}$ of the Stein algebra $\mathcal{O}(X)$ at the maximal ideal $m(x) := \{f \in \mathcal{O}(X) : f(x) = 0\}$ is noetherian — even more, it's excellent [2, p. 35]. According to a theorem of Mori-Nagata, the integral closure of a noetherian integral domain is completely normal [7, 4.3, 3.6], implying

$$(*) \qquad \mathcal{O}(\tilde{X}) \subset \bigcap_{x \in X} S_x^{-1} A .$$

For $f \in \mathcal{O}(\tilde{X})$, $a \in \tilde{X}$ and $b \in p^{-1}(p(a))$, it is now possible to conclude that $f(a) = f(b)$ is true. Let $x := \pi(a)$. Due to $(*)$, functions $g \in S_x$ and $h \in A$ exist with $f = h/g \circ \pi$. Since a and b are equivalent with respect to the equivalence relation R, $f(a) = f(b)$ follows, and a continuous function $F : Y \to \mathbf{C}$ exists with $F \circ p = f$.

Since the Stein complex structure on Y is not in general the canonical ringed quotient structure, it is still necessary to verify that F is holomorphic in order to prove the density of A in $\mathcal{O}(\tilde{X})$. To that end, let $H \in \mathcal{O}(Y)$ and $G \in \mathcal{O}(Y)$ have the property that $H \circ p = h$ and $G \circ p = g \circ \pi$. Such functions exist because $p^*(\mathcal{O}(Y)) = \bar{A}$ holds. Then $F = H/G$ follows, and the germ $F_{p(a)}$ is the germ of a holomorphic function at $p(a)$, since the germ $G_{p(a)}$ of G at $p(a)$ is a unit. The surjectivity of p implies that F is holomorphic on Y, completing the proof of the theorem.

Note that the topology induced by $\mathcal{O}(\tilde{X})$ on any subalgebra A of $\mathcal{O}(\tilde{X})$ is the metrizable topology of uniform convergence on compact subsets of X. Because the closure $\bar{A}$ of A in $\mathcal{O}(\tilde{X})$ is its completion, $\bar{A}$ can be obtained without referring directly to $\mathcal{O}(\tilde{X})$. Thus the Main Theorem can be stated as follows:

If $\tilde{X}$ denotes the normalization of an irreducible Stein space X, then $\mathcal{O}(\tilde{X})$ is the completion of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$.

3. Applications

In this section X will denote an irreducible Stein space with normalization $\pi : \tilde{X} \to X$, $\widetilde{\mathcal{O}(X)}$ will be the integral closure of the holomorphic functions $\mathcal{O}(X)$ on X, $\tilde{\mathcal{O}}(X)$ the Fréchet algebra of weakly holomorphic functions on X (or equivalently, the Fréchet algebra of holomorphic functions $\mathcal{O}(\tilde{X})$ on $\tilde{X}$), and

$$S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\} \quad \text{for} \quad x \in X .$$

Although the example given in the first section shows that the algebras $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ are not always equal, the inclusion $(*)$ in the proof of the Main Theorem implies that they are locally equal in the following sense.

THEOREM 2. *For every $x \in X$, the localizations of $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ with respect to S_x coincide.*

The next theorem implies an algebraic description of the topological closure of $\widetilde{\mathcal{O}(X)}$ in $\tilde{\mathcal{O}}(X)$.

THEOREM 3. $\mathcal{O}(\tilde{X}) = \bigcap_{x \in X} S_x^{-1} \widetilde{\mathcal{O}(X)}$.

Proof. Let $f \in M(\tilde{X}) = M(X)$ be such that for every $x \in X$ there is a $g \in S_x$ and an $h \in \widetilde{\mathcal{O}(X)}$, with $f = h/g \circ \pi$. Then the germ f_a of f at an arbitrary point $a \in \tilde{X}$ is holomorphic, because the germ of $g \circ \pi$ at a is a unit. Hence $f \in \mathcal{O}(\tilde{X})$, and the assertion is proved.

COROLLARY 2. The topological closure of $\widetilde{\mathcal{O}(X)}$ in $\tilde{\mathcal{O}}(X)$ is the intersection of the localizations of $\widetilde{\mathcal{O}(X)}$ with respect to S_x for all $x \in X$.

The next result characterizes the weakly holomorphic functions on X as being exactly those meromorphic functions on X which are almost integral over $\mathcal{O}(X)$.

COROLLARY 3. *$\mathcal{O}(\tilde{X})$ is completely normal.*

Proof. Let $f \in M(\tilde{X})$ be almost integral over $\mathcal{O}(\tilde{X})$. Then f is almost integral over $\mathcal{O}(X)$ and therefore over $S_x^{-1} \widetilde{\mathcal{O}(X)}$ for every $x \in X$ which has been shown to be completely normal in the proof of the Main Theorem. An application of Theorem 3 yields $f \in \mathcal{O}(\tilde{X})$ and hence the assertion.

Using the classical Oka-Weil-Cartan Theorem [1, Anhang zu VI, § 4], an immediate consequence of the Main Theorem is

THEOREM 4. *$\tilde{X}$ is $\widetilde{\mathcal{O}(X)}$-convex, $\widetilde{\mathcal{O}(X)}$-separable and has local coordinates by functions in $\widetilde{\mathcal{O}(X)}$.*

A property which ensures that the holomorphic functions on $\tilde{X}$ are integral over the holomorphic functions on X is that $\mathcal{O}(\tilde{X})$ is a finite $\mathcal{O}(X)$-module.

THEOREM 5. *Let $u \in \mathcal{O}(X)$ be any global universal denominator for X. Then $\mathcal{O}(\tilde{X})$ is isomorphic to the closed ideal $u\mathcal{O}(\tilde{X})$ in $\mathcal{O}(X)$, and $\mathcal{O}(\tilde{X})$ is a finite $\mathcal{O}(X)$-module if and only if this ideal is finitely generated.*

Proof. Recall that a global universal denominator u for X always exists [10, E.73a]. The multiplication map

$$\mathcal{O}(\tilde{X}) \to \mathcal{O}(X)\,,\ f \mapsto uf\,,$$

defines an injective $\mathcal{O}(X)$-module homomorphism. Thus, $\mathcal{O}(\tilde{X})$ is isomorphic to the ideal $u\mathcal{O}(\tilde{X})$ in $\mathcal{O}(X)$ which will now be denoted by I. Consider the transporter ideal $J := \tilde{\mathcal{O}} : \frac{1}{u}\mathcal{O}$ of $\frac{1}{u}\mathcal{O}$ into $\tilde{\mathcal{O}}$ which is a coherent sheaf of ideals in $\tilde{\mathcal{O}}$. The global sections $J(X)$ form a closed ideal of $\mathcal{O}(X)$ by a theorem of Cartan [4, 5], due again to the fact that X is Stein. Because $J(X) = I$ holds, the assertion follows.

COROLLARY 4. *If* $\mathcal{O}(\tilde{X})$ *does not coincide with* $\widetilde{\mathcal{O}(X)}$, *the closed ideal* $u\mathcal{O}(\tilde{X})$ *in* $\mathcal{O}(X)$ *is not finitely generated.*

In a Stein algebra $\mathcal{O}(X)$, every finitely generated ideal is closed, as Cartan [4, 5] showed. If X is at least two-dimensional, Forster [6] gave examples of closed ideals in $\mathcal{O}(X)$ which are not finitely generated. According to Corollary 4, the space constructed in § 1 gives a one-dimensional example.

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