

6. Coordinate free tensor calculus

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5. A PRIORI ESTIMATES: THE ORIGINAL WAY

According to proposition 4.3 we must prove now that, given any sequence of positive real numbers $(K_i), i \in \mathbb{N}$, there exists a sequence (C_i) such that

$$\forall i \in \mathbb{N}, \quad \| D^i P_\lambda(\varphi) \| \leq K_i$$

implies

$$\| \varphi \| \leq C_0, \quad \forall i \in \mathbb{N}, \quad \| D^i \nabla \bar{\nabla} \varphi \| \leq C_{i+2}.$$

These are *a priori* estimates of order zero, two, three, and so on In case $\lambda > 0$, the C^0 estimate is straightforward [2]. In case $\lambda = 0$, it becomes very tricky; proofs simpler than Yau's original one [24] (p. 352-359), based on the idea of uniformly estimating the $L^p(dX_g)$ norms of φ , may be found in [16] (dimension 2), [3] [21] and [4] (p. 148-149).

Estimates of order two and three are carried out by means of tensor calculus and of the Maximum Principle (for elliptic equations) [20] applied to *suitable* test functions. Though it is not everywhere clear in [21] [24], it is worth noting that the computations can be written intrinsically, i.e. without any reference to a *particular* system of coordinates (e.g. [2]), or even *coordinate free* (see section 6 below).

Further regularity is then recovered by Schauder theory e.g. [5] (lemma 1). In the sequel, we show how further estimates can be carried out instead, *just going ahead with coordinate free tensor calculus*. This occurs actually for any fully nonlinear second order elliptic equation on a compact Riemannian manifold, via a straightforward imitation of the device below.

Remark 5.1. It follows from the C^2 *a priori* estimates that the metrics g' are *a priori* uniformly equivalent to the original metric g (see e.g. [3], p. 75).

6. COORDINATE FREE TENSOR CALCULUS

Even coordinate free tensor calculus needs indices. Usually these indices refer to a *local* frame. Another way is to view these indices *globally* as labelling copies of the holomorphic and antiholomorphic tangent and cotangent bundles. From this point of view, a tensor written with indices is a section of the tensor product of a family of bundles indexed by an *unordered* set of indices (disregarding those indices subject to the summation convention).

We extend the summation convention as follows: we will be concerned only with lower indices. If a letter occurs twice, it refers to a contraction, which is taken with respect to g or to g' according to whether the letter occurs with a bar or with a prime. So,

$$\begin{aligned} T_{\dots a \dots \bar{a} \dots} &\text{ stands for } g^{a\bar{b}} T_{\dots a \dots \bar{b} \dots}, \quad \text{while} \\ T_{\dots a \dots a' \dots} &\text{ stands for } g'^{a\bar{b}} T_{\dots a \dots \bar{b} \dots}. \end{aligned}$$

As usual if $T_{a\dots l}$ is a tensor, further lower indices refer to covariant differentiation (with respect to g); so,

$$\begin{aligned} T_{a\dots lm} &\text{ stands for } \nabla_m T_{a\dots l}, \quad \text{while} \\ T_{a\dots l\bar{m}} &\text{ stands for } \bar{\nabla}_{\bar{m}} T_{a\dots l}. \end{aligned}$$

Our indices will be latin letters; greek letters will denote multi-indices. If α is a multi-index, $\bar{\alpha}$ will denote the *conjugate* multi-index (for instance if $\alpha = a\bar{b}c$, then $\bar{\alpha} = \bar{a}b\bar{c}$), while $|\alpha|$ denotes its length. We shall say that α is *mixed* if its length is at least two and, among the first two letters, *exactly* one has a bar.

The notations $D, \nabla, \bar{\nabla}, \parallel, \parallel$, were introduced in section 4.

Remark 6.1. Since covariant differentiation (with respect to g) and contraction with respect to g' *do not* commute, we observe that, for instance, the difference (recall $g' = g + \nabla\bar{\nabla}\phi$)

$$(3) \quad \phi_{aa'ab} - (\phi_{aa'\alpha})_b \equiv \phi_{ac\alpha} \phi_{a'c'b}$$

does not vanish.

7. HIGHER ORDER A PRIORI ESTIMATES: GENERALITIES

We want to prove by induction,

PROPOSITION 7.1. *Given $n \geq 4$, a sequence $(K_i), i \in \mathbf{N}$, and a finite sequence $C_0, \dots, C_{n-1}$, there exists C_n such that:*

$$\begin{aligned} \|\phi\| &\leq C_0, \quad \forall i = 0, \dots, n-3, \quad \|D^i \nabla \bar{\nabla} \phi\| \leq C_{i+2} \\ \text{and } \forall i \in \mathbf{N}, \quad \|D^i P_\lambda(\phi)\| &\leq K_i, \end{aligned}$$

implies

$$\|D^{n-2} \nabla \bar{\nabla} \phi\| \leq C_n.$$