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Autor: Mitchell, Stephen A.

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QUILLEN'S THEOREM ON BUILDINGS AND THE LOOPS ON A SYMMETRIC SPACE

by Stephen A. MITCHELL [1])

ABSTRACT

In the mid 70's Garland and Raghunathan, and (independently) Quillen discovered that ΩG (G a compact Lie group) is homotopy equivalent to an infinite dimensional flag variety, and that Bott's cell decomposition of ΩG can then be obtained as a Bruhat or Schubert cell decomposition. Quillen's method applies also to the loops on a compact symmetric space M, and involves identifying the path space of M with a certain Bruhat—Tits building. The details never appeared. In this paper we develop a theory of "topological" buildings and prove Quillen's theorem. We then show how one can rederive the Bott—Samelson theorems on ΩM, and the real and complex Bott periodicity theorems, from this point of view.

In the 1950's Bott, and Bott and Samelson, obtained a series of beautiful results on the topology of loop spaces of compact symmetric spaces: the Bott periodicity theorems [6], a cell structure (with various applications to homology) [4] [7], and a description of the Pontrjagin ring [5]. All of these theorems were proved using Morse theory. In the mid 70's another very different approach emerged in the work of Garland and Raghunathan [12] (who only consider the case of a compact Lie group) and (independently) Quillen [30]. The new point of view forms a part of the theory of loop groups: if G is a simply-connected compact Lie group, with complexification $G_{\mathbf{C}}$, the group $LG_{\mathbf{C}}$ of maps $S^1 \to G_{\mathbf{C}}$ can be regarded as an infinite dimensional complex algebraic group. The based loops ΩG then appear as a homogeneous space of $LG_{\mathbf{C}}$, analogous to a flag variety. If $M = G/K$ is a symmetric space, ΩM is a real form of ΩG. The cell structures of Bott and Samelson are obtained from a Bruhat decomposition of $LG_{\mathbf{C}}$, and their results can be derived from the combinatorics of the affine Weyl group. In addition, ΩM is the direct limit of finite dimensional "Schubert varieties", and recently this point of view has led to some new

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results on the homotopy type of $\Omega SU(n)$, $\Omega SU(n)/O(n)$, and $\Omega SU(2n)/Sp(n)$ ([9], [24], [26]).

There are two key ingredients. The first concerns the structure of the group of "algebraic loops" $L_{alg}G_{\mathbf{C}}$—i.e., the regular maps $\mathbf{C}^* \to G_{\mathbf{C}}$ (§ 3). Here the basic idea goes back to Iwahori and Matsumoto [16], following a suggestion of Bruhat. These authors show how to associate a Tits system (§ 2) to any Chevalley group over a local field, so that the Weyl group of the system is the affine Weyl group. Now if we write $\tilde{G}_{\mathbf{C}}$ for the group of points of the algebraic group $G_{\mathbf{C}}$ over $\mathbf{C}[z, z^{-1}]$, it is easy to see that $L_{alg}G_{\mathbf{C}} = \tilde{G}_{\mathbf{C}}$. Hence the results of [16] can be applied (at least after completing with respect to the ideal (z)) and we obtain a Tits system on $L_{alg}G_{\mathbf{C}}$. The group P of regular maps $\mathbf{C} \to G_{\mathbf{C}}$ is then a maximal parabolic subgroup, and the homogeneous space $\tilde{G}_{\mathbf{C}}/P$ (which is a direct limit of projective varieties) can be identified with $\Omega_{alg}G = \{f \in L_{alg}G_{\mathbf{C}}: f(S^1) \leqslant G \text{ and } f(1) = 1\}$ ("Iwasawa decomposition"). The axioms for a Tits system then yield a Bruhat or Schubert cell decomposition of $\tilde{G}_{\mathbf{C}}/P$, and hence a cell decomposition of $\Omega_{alg}G$. The cells are indexed by $\operatorname{Hom}(S^1, T)$, where T is a maximal torus. After some further technical work, this idea can be generalized to ΩM: if $M = G/K$, where $K = G^{\sigma}$ for some anti-complex involution σ on $G_{\mathbf{C}}$ preserving G, we can define an involution τ an $L_{alg}G_{\mathbf{C}}$ by $\tau f(z) = \sigma(f(\bar{z}))$. The fixed group is a real form of $\tilde{G}_{\mathbf{C}}$. Similarly, we replace $\Omega_{alg}G$ by $(\Omega_{alg}G)^{\tau}$—the space of $\mathbf{Z}/2$—equivariant loops—and the cell decomposition of $(\Omega_{alg}G)^{\tau}$ is obtained in an analogous way (§ 5). Of course to apply any of this to the original problem, we need the second ingredient: Let $\Omega_{alg}M = (\Omega_{alg}G)^{\tau}$, and note that ΩM can be identified with $(\Omega G)^{\tau}$.

THEOREM (Quillen). *The inclusion* $\Omega_{alg}M \to \Omega M$ *is a homotopy equivalence.*

In the case $M = G$ this theorem has several completely different proofs ([12], [29], [30]); it can also be deduced from Bott's work and the Bruhat decomposition [25]. The proof suggested by Quillen is particularly beautiful, and applies to all compact symmetric spaces M. The idea is the following, taking $M = G$ for simplicity: It is sufficient to produce a contractible space E on which $\Omega_{alg}G$ acts freely, with orbit space G. Quillen observes that a plausible candidate for E is already at hand. To any Tits system one can associate a certain simplicial complex (or space) $\mathscr{B}$—the *building*—and when the Weyl group of the system is infinite, as it is here, the building is contractible. In fact $\mathscr{B}$ is a certain quotient space of $L_{alg}G/T \times \Delta$, where Δ is a simplex of dimension equal to rank G. It follows that the $\Omega_{alg}G$ orbit space is a quotient of $G/T \times \Delta$. But it is a classical fact that $G = G/T \times \Delta/\sim$ (§ 1), and on inspection one sees that the two quotients are identical. In fact (this is also due to Quillen) $\mathscr{B}$ has a very concrete description: it is the space of paths in G of the form $f(e^{2\pi it}) \exp tX$, where $f \in \Omega_{alg}G$ and $X \in \mathfrak{g}$. The action of $\Omega_{alg}G$ on this space is obviously free, which completes the proof (§ 4).

The purpose of this paper is to give a detailed exposition of Quillen's idea, with reasonably complete proofs, and to show how one can derive the results of Bott and Samelson. (Along the way, we also give an axiomatic treatment of topological Tits systems.) The paper is organized as follows:

In § 1 we establish most of our notation concerning Lie groups, symmetric spaces, etc.; and collect some preliminary results. The most important point here is the classical description of M as a quotient of $K \times \Delta$, where Δ is the Cartan simplex. The reader will probably prefer to skim through this section first, and refer back to it later when necessary. The main references are [13], [22], and [33]; a short introduction to real forms, Satake diagrams, etc. can be found in [23].

In § 2 we discuss topological Tits systems (G, B, N, S) and their associated buildings. Although the axioms for a Tits system may seem obscure at first encounter, and lack the geometric appeal of Morse theory, it can not be denied that they are remarkably simple. The structure theory of such systems constitutes our main technical tool. However, in our context it is necessary to take into account the topology the system. We define topological Tits systems in a rather minimal way, and then state four additional axioms that will be satisfied by all the Tits systems considered in this paper. These axioms are fairly easy to verify in most cases, and suffice to establish various desirable properties: For example, that the Bruhat decomposition of a "flag space" G/P is a CW-decomposition, with the closure relations on the cells given by the Bruhat order on the Weyl group. Much of the treatment here is inspired by Steinberg ([32]) and Kac and Peterson ([17], [18], [19]). We then introduce the topological building $\mathfrak{B}_G$. It is a quotient space of $G/B \times \Delta$—in fact, it is precisely the homotopy colimit of the diagram of flag spaces $G/P_I (I \underset{\neq}{\subset} S)$. We show how to adopt the standard proofs of the Solomon-Tits theorem to the topological context. Thus $\mathfrak{B}_G$ is contractible if W is infinite and is a certain suspended quotient of G/B otherwise. As an example, we note that for the usual Tits system associated with a real form of a semisimple complex Lie group, the building can be identified with the "tangent cut locus" of the associated compact symmetric space.

In § 3 we briefly review some basic facts about algebraic loop groups. (See for example [1], [27] and [29] for details). The most important fact is that $LG_{\mathbf{C}}$ admits a suitable topological Tits system. The existence of the Tits system is proved more generally for Kac-Moody groups by Kac and Peterson [17], so we only sketch the proof.

In § 4 we prove Quillen's theorem on the building, in the case $M = G$. (We have separated this case from the general case in order to isolate the main idea, which is fairly simple.)

In § 5 we redo the results of § 3, 4 for a general M. Again, many of the more tedious technical results are only sketched. One key result is the existence of a suitable Bruhat decomposition of the real form $(L_{alg}G_{\mathbf{C}})^{\tau}$. Presumably this follows from the general theory of algebraic groups, but we have elected to give a direct proof that contains a result of some independent interest. The point is that the involution τ does not preserve the Iwahori subgroup $\tilde{B}$ (the "B" of the Tits system), so one can not simply apply τ to the $\tilde{B} - \tilde{B}$ double cosets in $\tilde{G}_{\mathbf{C}}$. However τ does preserve a certain parabolic $\tilde{Q}$ (canonically associated to the original involution σ), and hence preserves the $\tilde{Q} - \tilde{Q}$ double cosets. To analyze these, we show more generally that for any flag variety $G_{\mathbf{C}}/Q$ or $\tilde{G}_{\mathbf{C}}/\tilde{Q}$, the P-orbits (here P, Q are any parabolics) are holomorphic vector bundles over (finite dimensional) flag varieties of the Levi factor of P (which can be explicitly determined). This fact is certainly well known, but does not seem to appear in the literature. The details are banished to an appendix (§ 8). We also show in this section how to deduce various results from [7]: the cell structure on ΩM, the fact that these cells are all cycles mod 2 (or actual cycles, if M is of "splitting rank"), and the "somewhat mysterious" connection [7] between $H_*\Omega G$ and $H_*\Omega M$, when M is of maximal rank. (This connection becomes transparent in the present context.)

In § 6, we discuss six examples: $SU(2n)/Sp(n)$, $SU(n)/SO(n)$, $SO(2n)/U(n)$, $Sp(n)/U(n)$, S^n and $\mathbf{C}P^n$. Here, as elsewhere, we emphasize the way in which information can be obtained directly from the Satake and Dynkin diagrams.

In § 7, we reprove the real and complex periodicity theorems. In effect, we simply imitate Bott's original, beautiful proof, but with Morse theory replaced by topological Tits systems. The idea is that for certain commutator maps $K/H \xrightarrow{\varphi} \Omega G/K$, $\varphi(K/H)$ is a "Schubert subvariety", so the range of dimensions in which φ is an equivalence can be determined by merely counting cells. But as an added twist, we show that if one only considers the maps φ associated with the "miniscule roots" of M (these suffice for Bott periodicity), then this range of dimensions is not only determined by the root system (as Bott showed), but in fact can be read off directly, in a rather amusing way, from the Dynkin diagram. Thus the Bott periodicity theorems can be proved by inspecting the Dynkin diagrams of the classical symmetric spaces!

A traditional difficulty encountered by writers on this subject is the inordinate quantity of notation required: to the usual list of notations for root systems, Coxeter groups, complex Lie groups, etc., we must add still more notation for symmetric spaces, restricted root systems, loop groups, etc. Some further remarks: (1) we generally use a tilda for various "loop" analogues of classical objects, but this notation should be interpreted with care. For example, it $G_{\mathbf{C}}$ is a reductive complex algebraic group, $\tilde{G}_{\mathbf{C}}$ is the group of algebraic $G_{\mathbf{C}}$-valued loops; on the other hand, if B is a Borel subgroup of $\tilde{G}_{\mathbf{C}}$, its analogue is the Iwahori subgroup $\tilde{B}$—but $\tilde{B}$ is not a Borel subgroup, and is not the group of B-valued algebraic loops (see § 3). (2) in a similar vein, we generally use a subscript **R** to denote the analogue for a real form (given a fixed involution σ as above) of a complex object. For example, $G_{\mathbf{R}}$ is our real form of $G_{\mathbf{C}}$: $G_{\mathbf{R}} = (G_{\mathbf{C}})^{\sigma}$. On the other hand $B_{\mathbf{R}}$, the analogue of B, is usually called a "minimal parabolic". It is neither solvable nor connected in general, and does not equal B^{σ}, but nevertheless is the correct analogue of B (from the point of view of Tits systems). (3) Given a root system Φ (affine or ordinary), we frequently confuse, identify and otherwise comingle the following sets: (a) the simple roots (a system of positive roots having been fixed), (b) the simple reflections, (c) the nodes of the Dynkin diagram and (d) a set of integers $1, 2, \cdots l$ (*or* $0, 1, 2, \cdots l$) indexing all three of the above in a compatible way.

A final word on the origin of this paper: Quillen's work is unpublished, and, to the best of my knowledge, he never even circulated a manuscript. I first learned of the idea (of using the building) from a set of notes, kindly sent to me by Richard Kane, of a single lecture delivered by Quillen at MIT in July of 1975. Theorems 4.1, 4.2, 4.4 and 4.7 are stated there, and it is asserted that the methods and results carry over to symmetric spaces. The proofs of these theorems in the present paper are (for better or for worse) my own. The Bruhat and Iwasawa decompositions for algebraic loop groups (or at least their topological applications) are apparently due to Quillen and (independently) Garland and Raghunathan, although in their algebraic form these results go back to Iwahori and Matsumoto. The treatment here is largely based on work of Kac and Peterson [17]. Another approach is via the "Grassmanian model" representation of $\Omega_{alg}G$; this too is due to Quillen. We will not consider the Grassmanian model (or its obvious analogue for symmetric spaces, but see [9] for an example); there is a very thorough account of this approach in [29].

I would like to thank Suren Fernando for some very helpful conversations.

§ 1. Notation and Preliminaries

Except in § 2, G will always denote a compact connected Lie group of rank l; usually we will assume also that G is simple and simply-connected. Fix once and for all a maximal torus T in G, and let N denote the normalizer $N_G T$. The Weyl group W is N/T. Lie algebras are denoted as usual by Gothic letter: $\mathfrak{g}$, $\mathfrak{t}$, etc. To each G we can associate a reductive complex algebraic group $G_{\mathbf{C}}$—the complexification of G—with Lie algebra $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes \mathbf{C}$. It contains G as a maximal compact subgroup, and as the fixed group of an anti-complex involution. In fact $G \to G_{\mathbf{C}}$ defines an equivalence of categories (compact Lie groups) $\leftrightarrow$ (reductive complex algebraic groups).

$G_{\mathbf{C}}$ has a Borel subgroup (maximal connected solvable subgroup) B, unique up to conjugacy, which we can assume contains the Cartan subgroup (maximal algebraic torus) $T_{\mathbf{C}}$. There is a split extension $U \to B \to T_{\mathbf{C}}$ where U is the unipotent radical of B. There is also an opposite Borel subgroup B^- such that $B \cap B^- = T_{\mathbf{C}}$; it fits into a similar split extension $U^- \to B^- \to T_{\mathbf{C}}$. On the Lie algebra level we have $\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \mathfrak{u} \oplus \mathfrak{u}^-$, with $\mathfrak{u} \oplus \mathfrak{u}^-$ being precisely the sum of the nontrivial eigenspaces for the adjoint action of $\mathfrak{t}_{\mathbf{C}}$ on $\mathfrak{g}_{\mathbf{C}}$. The corresponding eigenfunctions $\lambda : \mathfrak{t}_{\mathbf{C}} \to \mathbf{C}$ map $\mathfrak{t}$ into $i\mathbf{R}$; as is customary we replace each λ by $\alpha = \lambda/2\pi i$ to obtain a set Φ of nontrivial $\mathbf{R}$-valued linear functionals on $\mathfrak{t}$-the real roots. These form a (reduced, crystallographic) root system in $\mathfrak{t}^*$. The positive roots Φ^+ correspond to $\mathfrak{u}$, the negative roots Φ^- to $\mathfrak{u}^-$. A simple system of roots $\alpha_1, \ldots, \alpha_l$ (here we assume G is semisimple of rank l) is then uniquely determined as the set of positive roots which are not decomposable as sums of positive roots. If we assume G is simple, so that Φ is irreducible, there is a unique "highest root" α_0, which is characterized by the property that for every positive root α, $\alpha_0 + \alpha$ is not a root. The corresponding eigenspace in $\mathfrak{u}$ is precisely the center of $\mathfrak{u}$. And, speaking of eigenspaces, let X_α denote the eigenspace (or "root subalgebra") of $\mathfrak{g}_{\mathbf{C}}$ associated to $\alpha \in \Phi$. For each α, the subalgebra of $\mathfrak{g}_{\mathbf{C}}$ generated by X_α and $X_{-\alpha}$ is isomorphic to $sl(2, \mathbf{C})$. The corresponding subgroup, isomorphic to $SL_2\mathbf{C}$ or $PSL_2\mathbf{C}$, is $G_{\mathbf{C},\alpha}$. Choosing generators E_α for the X_α, we obtain a basis for $\mathfrak{g}_{\mathbf{C}}$, consisting of the $E_\alpha (\alpha \in \Phi)$ and $H_\alpha = [E_\alpha, E_{-\alpha}]$ $(\alpha \in \Phi^+)$.

The basis above can be chosen so that the antilinear map $\mathfrak{g}_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}$ defined by $E_\alpha \to -E_{-\alpha}$ is a Lie algebra automorphism with fixed algebra $\mathfrak{g}$. In particular, then, we have $\mathfrak{g} = t \oplus (\oplus_{\alpha\in\Phi^+} Y_\alpha)$, where Y_α is spanned by $E_\alpha - E_{-\alpha}$ and $i(E_\alpha + E_{-\alpha})$. The Y_α are "eigenspaces" for the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$. Each Y_α generates a Lie algebra isomorphic to $\mathfrak{su}(2)$. The corresponding subgroups G_α, isomorphic to $SU(2)$ or $SO(3)$, are extremely important; for example, they generate G (if G is semisimple). Note G_α is a maximal compact subgroup of $G_{\mathbf{C},\alpha}$.

In $\mathfrak{t}$ there are three lattices: the coroot lattice R, spanned by the coroots $\alpha^\vee = 2\alpha/\alpha\cdot\alpha$ ($\mathfrak{t}$ is identified with $\mathfrak{t}^*$ via a W-invariant inner product), the integral lattice $I = Ker(\exp: \mathfrak{t} \to T)$, and the coweight lattice $J = \{X \in \mathfrak{t}: \alpha(x) \in \mathbf{Z}\ \forall\, \alpha \in \Phi\}$. We have $R \leqslant I \leqslant J$, with $I/R \cong \pi_1 G$ and $J/I \cong C(G)$. If we think of R as a group of isometries (translation) of $\mathfrak{t}$, then R is normalized by W; the *affine Weyl group* $\tilde{W}$ is the semidirect product RW. Next, consider the Stiefel diagram, which consists of the hyperplanes $P_{\alpha,n} = \{X \in t: \alpha(x) = n\}$ ($\alpha\in\Phi$, $n\in\mathbf{Z}$). The connected components of the complement of the diagram are the *alcoves*, and we have:

(1.1) THEOREM. *(a) $\tilde{W}$ acts simply transitively on the alcoves; (b) $\tilde{W}$ is generated by the reflections in the walls of any fixed alcove.* □

Now let $\mathscr{C}^+$ be the positive Weyl chamber: $\{X \in t: \alpha(x) > 0\ \forall\, \alpha \in \Phi^+\}$. Assume (for convenience) that G is simple. Then as our standard alcove we take $\mathscr{A}^+ = \{X \in \mathscr{C}^+ : \alpha_0(X) < 1\}$. The closure Δ of $\mathscr{A}^+$ is an l-simplex—the *Cartan simplex*; its walls are the hyperplanes $\alpha_i = 0 (1 \leqslant i \leqslant l)$, $\alpha_0 = 1$. The wall $\alpha_0 = 1$ will be called the *outer wall.* Thus $\tilde{W}$ is generated by the set $\tilde{S} = S \cup \{s_0\}$, where s_0 is reflection in the outer wall. For each subset I of $\tilde{S}$ the I-face Δ_I of Δ is defined by $\Delta_I = \{X \in \Delta: \alpha_i(x) = 0$ if $i \in I$, $i \neq 0$, $\alpha_0(x) = 1$ if $0 \in I\}$. (Here $\tilde{S} = \{s_0, \cdots, s_l\} \equiv \{0, 1, \cdots l\}$). We let $\mathring{\Delta}_I$ denote the interior of Δ_I, so that Δ is the disjoint union of the $\mathring{\Delta}_I$. The isotropy group in $\tilde{W}$ of any $X \in \mathring{\Delta}_I$ is precisely $\tilde{W}_I$ (the subgroup generated by I).

(1.2) THEOREM. *Suppose $X, Y \in \Delta$ and $\sigma X = Y$ for some $\sigma \in \tilde{W}$. Then $X = Y$ and $\sigma \in \tilde{W}_I$, where $I = \{s \in \tilde{S}: sX = X\}$.* □

The most important feature of Δ, for our purposes, is the following:

(1.3) THEOREM. *Every element of G is conjugate to $\exp X$ for some $X \in \Delta$. If G is simply-connected, X is unique.*

[The proof of this classical theorem is easily obtained from what we have stated so far, together with the conjugacy of maximal tori and the fact that two elements of T conjugate in G are conjugate by an element of W]. □

The first part of (1.3) asserts that the map $G/T \times \Delta \xrightarrow{\pi} G$ given by $\pi(gT, X) = g \exp X\, g^{-1}$ is surjective. Thus G is a quotient space $G/T \times \Delta/\sim$ for a certain equivalence relation $\sim$. If G is simply-connected, the second part asserts that the equivalence relation is given by $(g_1T, X_1) \sim (g_2T, X_2)$ if and only if $X_1 = X_2 = X$ (say), and $g_1 = g_2 \bmod C_G \exp X$. Now $C_g \exp X(\{Y \in g: (\exp X) \cdot Y = Y\})$ is easily determined (we write $g \cdot X$ for $(\mathrm{Ad} g)(X)$): $C_g \exp X = (\oplus_{\alpha(x) \in \mathbf{Z}} V_\alpha) \oplus t$, and furthermore $\{\alpha \in \Phi: \alpha(X) \in \mathbf{Z}\}$ is generated by the simple roots it contains—provided that $(-\alpha_0)$ is counted as a simple root. (Of course for $X \in \Delta$, $\alpha(x) \in \mathbf{Z}$ means $\alpha(x) = 0, \pm 1$). In other words, if $X \in \mathring{\Delta}_I$, the identity component of $C_G \exp X$ is the (closed) subgroup G_I generated by T and the G_{α_i}, $i \in I$. We recall here that although centralizers of tori are always connected, centralizers of elements need not be. Fortunately, however, there is the following result.

(1.4) THEOREM (Borel [2], Bott [unpublished]). *If Θ is an automorphism of a simply-connected compact Lie group G, the fixed group of Θ is connected.* □

In particular centralizers are connected in this case, so $C_G \exp X = G_I$. We summarize the preceeding discussion in the next theorem.

(1.5) THEOREM. *Let G be a simple, simply-connected compact Lie group, regarded as a quotient space of $G/T \times \Delta$ as above. Then the equivalence relation on $G/T \times \Delta$ is given by $(g_1T, X) \sim (g_2T, X)$ if $X \in \mathring{\Delta}_I$ and $g_1 = g_2 \bmod G_I$.* □

We turn next to symmetric spaces. Let σ be an involution of a semi-simple G with fixed group K, and let K' be any subgroup of K containing the identity component. For our purposes a symmetric space is by definition a space of the form G/K'. However we will consider exclusively simply-connected symmetric spaces; in that case K' is necessarily connected. Lifting σ to an involution $\tilde{\sigma}$ of the universal cover $\tilde{G}$ of G, we see that $G/K' = \tilde{G}/K''$, where K'' is the fixed group of $\tilde{\sigma}$. Hence we may assume without loss of generality that G itself is simply-connected, and in that case the Borel-Bott theorem guarantees that K is connected. The induced involution on $\mathfrak{g}$ will also be denoted by σ. We have $\mathfrak{g} = k \oplus m$, where $\mathfrak{m}$ is the (-1)-eigenspace of σ. Let $M = \exp m$. Then:

(1.6) THEOREM. *The map* $\eta: G/K \to M$ *given by* $\eta(gk) = g\sigma(g^{-1})$ *is a* K*-equivariant homeomorphism.* (K *acts on* M *by conjugation.*) □

From now on we identify G/K with M. Let $\mathfrak{t}_{\mathfrak{m}}$ be a maximal abelian subspace of $\mathfrak{m}$ (any two such are K-conjugate); we can assume $\mathfrak{t}_{\mathfrak{m}} \subset t$. The torus $T_m = \exp \mathfrak{t}_m$ is a *maximal torus* of M (or of G/K). The *relative Weyl group* $W_{G,K}$ is $N_K t_m / C_K t_m$; as in the absolute case, it is a finite group.

Now the involution σ on $\mathfrak{g}$ (resp. G) extends uniquely to an anti-complex involution on $\mathfrak{g}_{\mathbf{C}}$ (resp. $G_{\mathbf{C}}$). Passing to fixed points, we obtain the associated real forms $G_{\mathbf{R}} = (G_{\mathbf{C}})^{\sigma}$ (not to be confused with $(G^{\sigma})_{\mathbf{C}}$!) and $\mathfrak{g}_{\mathbf{R}} = (\mathfrak{g}_{\mathbf{C}})^{\sigma}$. $G_{\mathbf{R}}$ is semisimple real Lie group, containing K as a maximal compact, and will play an important role.

Up to conjugacy, we can assume that σ is in "normal form": σ preserves $\mathfrak{t}_{\mathbf{C}}$, and commutes with the "compact" involution of $\mathfrak{g}_{\mathbf{C}}$ (the involution with fixed algebra $\mathfrak{g}$). With this assumption, we now consider the associated *relative root system*. Since σ is antilinear, its action on $\mathfrak{t}_{\mathbf{C}}^*$ is given by $(\sigma\lambda)(x) = \overline{\lambda(\sigma x)}$. This action permutes the complex roots, and yields an involution on the real roots $\Phi: (\sigma\alpha)(x) = -\alpha(\sigma x)$. Let Φ_0 denote the set of roots which restrict to zero on $\mathfrak{t}_m$; and let W_0 denote the associated Weyl group (note Φ_0 is spanned by the simple roots it contains; W_0 is the subgroup generated by the corresponding simple reflections). The *relative root system* Σ is the set of nonzero linear functionals β on $\mathfrak{t}_m$ which are restrictions of roots $\alpha \in \Phi$. One can show that Σ is indeed a root system, although it is not necessarily reduced—i.e., there may be roots β such that 2β is also a root. The following result is due to Satake [31]:

(1.7) THEOREM. *There is a base* B *(simple system of roots) for* Φ *such that if* Φ^+ *is the corresponding set of positive roots,* σ *preserves* $\Phi^+ - \Phi_0$. *Furthermore any such base satisfies (a)* $B \cap \Phi_0$ *is a base for* Φ_0 *and (b) For each* $\alpha \in B - \Phi_0$, *there is a unique* $\alpha' \in B - \Phi_0$ *such that* $\sigma\alpha = \alpha' \bmod \mathbf{Z}\Phi$. □

Using this theorem, the *Satake diagram* of G/K can be described as follows. Start with the Dynkin diagram of G; its nodes are labelled by the simple roots of Φ (or by the set S). Color the nodes belonging to Φ_0 black and color the remaining nodes white. By part (b) there is an involution (possibly trivial) on the set of white nodes; this is indicated by drawing double arrows $\leftrightarrow$ between the nodes of each nontrivial orbit. Six examples are given in § 6; see [13], pp. 532-4 for a list of all possible Satake diagrams. To capture all of the structure of G/K another diagram is needed, which we will call the Dynkin Diagram of G/K. First define the *multiplicity* m_β of a root β in Σ to be the number of roots in Φ which restrict to β. Then the Dynkin diagram of G/K is the Dynkin diagram of Σ with the nodes labelled by their multiplicities; if β is a simple root such that 2β is also a root, the β-node is to be labelled by $(m_\beta, m_{2\beta})$. Again, see § 6 for examples; for the moment we just mention an extreme case: If G/K has maximal rank—i.e. $\mathfrak{t}_m = \mathfrak{t}$—then $G_\mathbf{R}$ is the so-called split real form of $G_\mathbf{C}$. The nodes of the Satake diagram are then all white, with trivial involution, $\Phi = \Sigma$ and $\mathfrak{m}_\alpha = 1$ for all α. For example, take $G = SU(n)$, $\sigma(A) = \bar{A}$, $K = SO(n)$ and $G_\mathbf{R} = SL(n, \mathbf{R})$. (The opposite extreme —all nodes on the Satake diagram black—corresponds to the compact involution on $G_\mathbf{C}$ (*so* $\sigma\,|_G = 1$), and will be ignored.)

For our purposes it is necessary to consider the extended Satake and Dynkin diagrams. We recall here that the extended Dynkin diagram of an irreducible (reduced) root system is obtained formally by considering $-\alpha_0$ as a simple root and adjoining a corresponding node to the ordinary Dynkin diagram. (For us this definition is motivated by loop groups (§ 3), but it has many other uses—for example, in the Borel-de Siebenthal classification of maximal rank subgroups of G [3]). Now in view of (1.7) it is clear that σ_0 restricts to the highest root of Σ, and so in particular restricts non-trivially. Hence the extended Satake diagram is obtained by coloring the $(-\alpha_0)$-node white (and leaving it fixed under the involution, for reasons which should become clear later). The extended Dynkin diagram for G/K is obtained from the ordinary one by adjoining $-\alpha_0$ and labelling it by its multiplicity ($2\alpha_0$ is never a root).

Next, we will need the analogues of the subgroups $G_{\mathbf{C},\alpha}$ and G_α in the real form $G_\mathbf{R}$. Let β be a simple root in Σ, and let I_β be the subset of S determined as follows (cf. [22], pp. 135-36): In the Satake diagram form the subdiagram consisting of the black nodes and the set of white nodes that correspond to β under restriction (there are either one or two such white nodes). Then, in this subdiagram, take the path component that contains the white node(s) (even when there are two white nodes, they lie in one component). The nodes of the diagram obtained define the set I_β of simple roots in Φ. The subgroup G_{I_β} of G is preserved by σ, as is its commutator subgroup G'_{I_β}, and the fixed group $K_\beta = (G'_{I_\beta})^\sigma$ is the desired analogue of G_α. Similarly, $G_{\mathbf{R},\beta}$ is the σ-fixed group in $(G_\mathbf{C})_{I_\beta}$. Note that we have selected a sub—Satake diagram corresponding to the rank one symmetric space G_{I_β}/K_β.

Examples. In the split case, identifying Φ with Σ, we have $G_{I_\beta} = G_\beta \cong SU(2)$ and $K_\beta \cong SO(2)$ for all β. For the usual involution on $SU(2n)$ with $K = Sp(n)$, the subdiagrams obtained all have the form ●——○——●, so $G_{I_\beta} \cong SU(4)$ and $K_\beta \cong Sp(2)$ for all β (§ 6.1).

If β_0 is the highest root of Σ, K_{β_0}, $(G_0)_{\beta_0}$ are similarly defined, using the extended Satake diagram.

Lattices are defined exactly as before, using t_m, T_m and Σ in place of t, T, Φ. The coroot, integral, and coweight lattices for M will be denoted R_m, I_m, J_m, respectively. In fact, in each case the lattice for M is obtained by simply intersecting the corresponding lattice for G (in t) with t_m. The definition of the affine Weyl group $\tilde{W}_{G,K}$, the Stiefel diagram, alcoves, Cartan simplex Δ_m etc. are exactly as above—indeed these depend only on the root system Σ. In fact $\Delta_m = \Delta \cap t_m$. Theorems (1.3) and (1.5) also go through in the following form, for example.

(1.8) THEOREM. *Let G be a simple compact Lie group with involution σ and fixed group K as above. Then every element of M is K-conjugate to an element of the form $\exp X, X \in \Delta_m$. If G/K is simply-connected, X is unique.*

To state the analogue of (1.5), we need to determine $C_K \exp X$ for $X \in (\mathring{\Delta}_m)_I$. Here I is a subset of $\tilde{S}_{\mathbf{R}}$—the set of simple roots of Σ. Clearly $C_K \exp X = (C_G \exp)^\sigma$. It follows easily that $C_K \exp X = (G_{I'})^\sigma$, where I' is obtained from I in the obvious way: In the extended Satake diagram, I' corresponds to the black nodes together with all the white nodes that "restrict" to the nodes of I. (For example, if I is the empty set—i.e., X lies in the interior of the Cartan simplex Δ_m – I' corresponds to the black nodes and $C_K \exp X = (G_{I'})^\sigma = C_K t_m$). Let $K_I = (G_{I'})^\sigma$.

(1.9) THEOREM. *Let $G, \sigma, K,$ be as in (1.8), with $G/K \equiv M$ simply-connected, and regard M as a quotient space of $K/C_K t_m \times \Delta_m$ via the map $(kC_K T_m, X) \mapsto k \exp X\, k^{-1}$. Then the equivalence relation on $K/C_K t_m \times \Delta_m$ is given by $(k_1, X) \sim (k_2, X)$ if $X \in (\mathring{\Delta}_m)_I$ and $k_1 = k_2 \bmod K_I$.*

The final volley in our barrage of notation has to do with Weyl groups. If (W, S) is any Coxeter system, and I is a subset of S, W_I is the subcoxeter system generated by I. Each coset wW_I has a unique element X of minimal length, and $l(xy) = l(x) + l(y)$ for all $y \in W_I$ ($l(w)$ is the length of w as a word in the elements of S). We let W^I denote the set of such minimal length elements. We also recall that W^I has a partial order—the Bruhat order—defined by setting $x \leqslant y$ if y has a reduced decomposition

$y = s_1 \cdots s_k (s_i \in S)$ and x has a reduced decomposition obtained by deleting some subset of the s_i's occuring in y. (For a very nice account of these related matters, see [14]). If W is finite, W has a unique element w_0 of maximal length, we define the length of W to be $l(w_0)$.

§ 2. Topological Buildings

A *Tits system* (G, B, N, S) consists of a group G, subgroups B and N, and a set S, which satisfy the following axioms:

(2.1) $B \cap N$ is normal in N, and S is a set of involutions generating $W \equiv N/B \cap N$,

(2.2) B and N generate G,

(2.3) If $s \in S$, $sBs \neq B$,

(2.4) if $s \in S$, $w \in W$, then $sBw \leqslant BwB \cup BswB$.

(The use of expressions such as sBw is a standard abuse of notation).

Example. Let G be a reductive algebraic group over an algebraically closed field (e.g., $GL(n, \mathbf{C})$), let B be a Borel subgroup (e.g. upper triangular matrices), and let N be the normalizer of a maximal torus (that lies in B). This data determines a set S of simple reflections generating the Weyl group W (e.g., the usual generators $s_1, \ldots, s_{n-1}$ of Σ_n). Then one of the main results in the structure theory of reductive groups is that (G, B, N, S) is a Tits system (see for example [15]).

Throughout this paper we will assume that the set S is finite; its cardinality l is the *rank* of the system.

We next list some of the important properties of a Tits system.

(2.5) (Bruhat Decomposition) $G = \coprod_{w \in W} BwB$ (disjoint union),

(2.6) (W, S) is a Coxeter system.

A subgroup P of G is *parabolic* if it contains a conjugate of B. In particular if $I \subseteq S$, the subgroup P_I generated by B and I is parabolic.

(2.7) (a) The parabolic subgroups containing B are precisely the P_I, $I \subseteq S$. No two of these are conjugate; in particular there are exactly 2^l such subgroups, which form a lattice isomorphic to the lattice of subsets of S.

(b) $P_I = BW_IB$

(c) Every parabolic P is self-normalizing: $N_G P = P$.

(2.8) (Bruhat decomposition, general version) $G = \coprod_{w \in W_I \backslash W / W_J} P_I w P_J$ (disjoint union).

The next result, which we will refer to as the *Steinberg Lemma,* is somewhat technical; however it is not hard to prove and is extremely useful. It is a mild generalization of Theorem 15 of [32] and Proposition 3.1 of [19].

(2.9) Let $I \subseteq S$ and suppose w is the unique element of minimal length of wW_I. Suppose $w = w_1 \ldots w_k$ where $l(w) = l(w_1) + \ldots + l(w_k)$. Then

(a) If Y_i is any subset of Bw_iB such that $Y_i \to Bw_iB/B$ is bijective (resp. surjective) $(1 \leqslant i \leqslant k)$, then $Y_1 \times Y_2 \ldots \times Y_k \to BwP_I/P_I$ is bijective (resp. surjective).

(b) Suppose $w_i \in S$, $1 \leqslant i \leqslant k$ i.e., $w_1 \ldots w_k$ is a reduced decomposition of w). Let Z_i, $1 \leqslant i \leqslant k$, be any subset containing 1 of P_{w_i} such that $Z_i \to P_{w_i}/B$ is surjective. Then the image of $Z_1 \times \cdots Z_k \to G/P_I$ is $\coprod_{x \leqslant w} BxP_I/P_I$.

The maps in (a), (b) are the obvious multiplication/projection maps. Part b refers to the Bruhat order on W^I.

(2.10) *Remark.* The Tits system of a reductive algebraic group has several additional features: $B = HU$, where H is a maximal torus and U is a normal unipotent subgroup, U in turn is described in terms of its root subgroups, and there is an "opposite" Borel subgroup B^- such that $B \cap B^- = H$. This additional structure can also be axiomatized in an elegant way, leading to the "refined" Tits system of Kac and Peterson [19]. One then obtains, for example, the *Birkhoff decomposition* $G = \coprod_{w \in W} B^- wB$ as a consequence of the axioms.

We now define a *topological* Tits system to be a Tits system such that G is a topological group, B and N are closed subgroups, and W is discrete (i.e. $N \cap B$ is an open subgroup of N). We will usually also assume (for reasons which will be apparent shortly):

(2.11) *Axiom.* If I is a proper subset of S, W_I is finite.

This axiom is satisfied if W is an irreducible affine Weyl group, or finite. To get any interesting results some further axiom seems necessary. One direction is considered in [11], where the groups in question are algebraic groups over local fields, with the valuation topology. Here, with loop groups in mind, the following axiom seems efficient:

(2.12) *Axiom.* For each $s \in S$ there is a subset A_s of P_s such that (a) $A_s B = P_s$, (b) A_s is compact and contains 1, and (c) $A_s = \overline{A_s \cap BsB}$. This axiom is motivated by Steinberg's approach [32].

(2.13) PROPOSITION. *Let* (G, B, N, S) *be a topological Tits system satisfying (2.12). Then*

(a) $\overline{BwB} = \coprod_{x \leqslant w} BxB (w \in W)$. *More generally if* $I \leqslant S$, *and* $w \in W^I$, $\overline{BwP_I} = \coprod_{x \leqslant w} BxP_I$ *(here* $x \in W^I$*),*

(b) *B-orbits in* G/P_I *are locally closed,*

(c) *If* W *satisfies (2.11), parabolic subgroups are closed.*

Proof. First we show $P_s = \overline{BsB}$. Since $P_s = A_s B$, with A_s compact and B closed, P_s is closed, so $P_s \geqslant \overline{BsB}$. But also $B \subset P_s = A_s B \subset \overline{BsB}$, which proves our claim. Part (a) now follows easily from the Steinberg lemma: Let $M_w = \coprod_{x \leqslant w} BxP_I$, and let $w = s_1 \cdots s_k$ be a reduced decomposition. Then $M_w = A_1 \cdots A_k P_I$ and hence is closed. Next, suppose $x \leqslant w$; we must show $BxB \leqslant \overline{BwB}$. It is enough to consider the case when X has a reduced decomposition $x = s_1 \cdots \hat{s}_i \cdots s_k$ (omit s_i). Then

$$BxP_I = A'_1 \cdots A'_{i-1} A'_{i+1} \cdots A'_k P_I \leqslant A'_1 \cdots A'_{i-1} \bar{A}_i \cdots A'_k P_I \leqslant \overline{BwP_I}$$

(since $1 \in A_i$), where $A'_i = A_i \cap Bs_iB$. This proves (a). Part (b) is immediate since the complement of BwP_I in its closure is a finite union of sets of the form M_x, hence is closed. Since $P_I = BW_IB$, (c) is also immediate from (a) if W_I is finite. □

From now on we will assume 2.11 and 2.12. The homogeneous spaces G/P_I will be called *flag spaces.* The B-orbits $E_w = BwP_I/P_I$ are *Schubert strata* and the compact subspaces $\overline{E_W}$ are *Schubert subspaces.*

We next consider the *building* $\mathscr{B}_G$ associated to a topological Tits system (G, B, N, S). (The notation is ambiguous—indeed in the case of loop groups, G will support two natural but totally different Tits system. However the system we have in mind will be clear from the context.) In the discrete case, $\mathscr{B}_G$ is usually defined as the following simplicial complex. The vertices are the maximal (proper) parabolics, and $P_1 \cdots P_k$ span a simplex if $\bigcap_{i=1}^{k} P_i$ contains a conjugate of B. In general it is convenient to reinterpret this definition as follows: first of all, by definition every parabolic P is conjugate to a unique P_I; we say that P has type I. Thus the maximal parabolics are the parabolics of type $[s]$, where $[s] = S - \{s\}$. More generally the k-simplices correspond to the parabolics of type I, where $|I| = l - k - 1$. Thus the simplices all have dimension $\leqslant l - 1$, with the $l - 1$ simplices corresponding to the conjugates of B. Furthermore, in view of 2.7 (c), the set of parabolics of type I is canonically identified with $G/P_I - xP_I$ corresponding to xP_Ix^{-1}. One can casily check that with this interpretation, a simplex xP_I is a face of a simplex yP_J if and only if $I \supset J$ and $xP_I = yP_I$. In particular, every simplex is a face of some $l - 1$ simplex. Hence, as a set, B_G can be identified with $G/B \times \Delta/\sim$, where Δ is the $l - 1$ simplex with vertex set S, and $(g_1B, X_1) \sim (g_2B, X_2)$ if $X_1 = X = X_2, X \in \Delta_I$, and $g_1P_I = g_2P_I$. (Here Δ_I is the face of Δ corresponding to $I \leqslant S$.) We will therefore *define* the building $\mathscr{B}_G$ associated to the topological Tits system (G, B, N, S) to be $G/B \times \Delta$ modulo this equivalence relation, with the quotient topology.

Remark. Another way of expressing this is as follows: Let C be the category defined by the poset of proper subsets of S (including the empty set). We have a functor from C to topological spaces given by $I \mapsto G/P_I$. Then $\mathscr{B}_G$ is precisely the homotopy colimit of this diagram of spaces, in the sense of [8], p. 327 ff.

(2.14) PROPOSITION. *The equivalence relation on* $G/B \times \Delta^{l-1}$ *is generated by the relations* $(g_1B, X) \sim (g_2B, X)$ *if* X *lies on the wall* Δ_s *and* $g_1P_s = g_2P_s$.

Proof. In the usual language, (2.14) is the assertion that any two chambers are linked by a "gallery". (See e.g. [11], appendix.) Since the action of G on G/B induces a well defined action on $\mathscr{B}_G$, we are reduced to showing that if $(B, X) \sim (gB, X)$—i.e. $X \in \Delta_I$ and $g \in P_I$—then (B, X) and (gB, X) are linked by a sequence of relations of the stated type. But $gB = bwB$ with $w \in W_I$; hence if $w = s_1 \cdots s_k$ is a reduced decomposition, the elements $(B, X), (bs_1B, X), (bs_1s_2B, X), \ldots (bwB, X)$ provide the desired sequence. □

Note that the set Δ is a fundamental domain for the action of G on $\mathscr{B}_G$. On the other hand, it is easy to check that the closed subspace $\mathscr{B}_W$ consisting of the pairs $(wB, X), w \in W$, is a fundamental domain for the B action. (The point is that if $bw_1P_I = w_2P_I$, then $w_1P_I = w_2P_I$, by the Bruhat decomposition.) This space $\mathscr{B}_W$, which we will call the *foundation* of the building, is a simplicial complex since W is discrete. Since it will turn out that $\mathscr{B}_G$ is in a sense a "thickening" of the foundation, the following well known description of $\mathscr{B}_W$ may be of interest.

(2.15) PROPOSITION. *Suppose* Φ *is an irreducible root system in the Euclidean space* V. *Then*

(a) *If* W *is the affine Weyl group associated to* Φ, *then* $\mathscr{B}_W$ *is isomorphic as a simplicial* W*-complex to* V *(triangulated by the hyperplanes of* Φ*).*

(b) *If* W *is the Weyl group of* Φ, $\mathscr{B}_W$ *is isomorphic as simplicial* W*-complex to the unit sphere of* V, *triangulated by the Weyl chambers. More precisely,* $\mathscr{B}_W$ *can be identified with the* W *orbit of the outer wall of the Cartan simplex.*

Proof. For (a), map $W \times \Delta \xrightarrow{\varphi} V$ by identifying Δ with the Cartan simplex in V and using the action map. Then φ is onto (1.1) and furthermore $\varphi(w_1, x) = \varphi(w_2, X_2)$ if and only if $X_1 = X = X_2, X \in \Delta_I$, and $w_1 = w_2$ modulo the isotropy group of X. But this isotropy group is precisely W_I (1.2), so φ factors through the desired isomorphism $\mathscr{B}_W \to V$. The proof of (b) is similar. □

We now come to the main result of this section. Filter G/B by $F_k(G/B) = \coprod_{l(w) \leqslant k} E_w$. Similarly, $\mathscr{B}_G$ is filtered by $F_k(\mathscr{B}_G) = F_k(G/B) \times \Delta/\sim$.

(2.16) THEOREM. *Let* (G, B, N, S) *be a topological Tits system which either is discrete or satisfies (2.11) and (2.12). Assume also that the inclusions* $F_k(B_G) \subset F_{k+1}(B_G)$ *are cofibrations. Then*

(a) *If* W *is infinite,* $\mathscr{B}_G$ *is contractible.*

(b) *If* W *is finite of length* $r, \mathscr{B}_G$ *is homotopy equivalent to the* $(l-1)$ *st suspension* $S^{l-1} \wedge (F_r(G/B))/F_{r-1}(G/B))$.

Remark. If G is discrete, $F_k\mathscr{B}_G$ is a subcomplex of the simplicial complex $\mathscr{B}_G$, so the cofibration hypothesis is automatically satisfied. Furthermore if W is finite the smash product in (b) is just a wedge of $|F_rG/B - F_{r-1}G/B|$ $(l-1)$-spheres. This case is due to Solomon and Tits; cf. [11].

Proof of (2.16). Let X_k denote $F_k\mathscr{B}_G/F_{k-1}\mathscr{B}_G$, and let $X'_k = F_k(G/B)/F_{k-1}(G/B)$. Then we will show

(2.17) If k is less than the length of W, X_k is contractible. If $k = r$ = length of W, X_k is homeomorphic to $(F_r(G/B)/F_{r-1}(G/B) \wedge S^{l-1})$.

If W is infinite, it follows that $F_k\mathscr{B}_G$ is contractible for all k, and hence $\mathscr{B}_G$ is contractible. If W is finite, part (b) of the theorem is also immediate.

To prove 2.17, first consider the quotient map $\pi: F_k(G/B) \times \Delta \to X_k$. In fact π is merely collapsing a subspace to a point:

(2.18) Let $A_1 = (b_1 w_1 B, X_1)$, $A_2 = (b_2 w_2 B, X_2)$. If $\pi(A_1) = \pi(A_2)$, then either $A_1 = A_2$ or $\pi(A_1) = \pi(A_2) = *$ ($*$ is the basepoint $F_{k-1}B_G$).

For suppose $\pi(A_1) \neq *$, and $X_1 = \mathring{\Delta}_I$. Then $l(w_1) = k$ and $w_1 \in W^I$. This forces $X_1 = X_2$ and $w_1 = w_2 \bmod W_I$; hence $w_1 = w_2$ since $l(w_2) \leqslant k$ by assumption. Then $b_1 w_1 P_I = b_2 w_1 P_I$. But whenever $w \in W^I, b_1 w P_I = b_2 w P_I$ implies $b_1 w B = b_2 w B$ (easy exercise).

It now follows that $X_k = \bigvee_{l(w)=k} X_w$, where X_w is the image of $\bar{E}_w \times \Delta$ in X_k, and to prove (2.17) we need only consider a fixed X_w. Let $X'_w = \bar{E}_w/(\bar{E}_w - E_w)$, and let Δ' be the subcomplex of Δ consisting of the walls Δ_s such that $l(ws) < l(w)$. Then (2.18) implies:

(2.19) $X_w = X'_w \wedge (\Delta/\Delta')$.

For X_w is $\bar{E}_w \times \Delta$ modulo the subspace of points which are equivalent (in $\mathscr{B}_G$) to a point of lower filtration, namely, $\bar{E}_w \times \Delta' \cup \bar{E}_w - E_w \times \Delta$. It remains to identify Δ'. Since $F_0\mathscr{B}_G = \Delta$ is contractible, we may assume $k \geqslant 1$; then Δ' is nonempty. If $k < l(W)$, then there is at least one $s \in S$ such that $l(ws) > l(w)$; hence Δ' is not the entire boundary of Δ and Δ/Δ' is contractible. If $k = l(W)$, then w is unique, $\Delta' =$ boundary of Δ, and $\Delta/\Delta' = S^{l-1}$. This completes the proof of (2.17), and of the theorem. □

Remark. Our proof of Theorem 2.16 is an adaptation of the standard (discrete) proof to the topological setting. Much of the proof depends only on the Weyl group W, and indeed shows e.g. for W infinite that the foundation of the building is contractible. In fact the deformation of $F_k(\mathscr{B}_W)$ into $F_{k-1}(\mathscr{B}_W)$ has the property that the isotropy group in B of a point X in $\mathscr{B}_W$ is an increasing function of time, and hence extends uniquely to a B-equivariant deformation of $F_k(B_G)$. In the discrete case this extension is automatically continuous, and shows that Theorem (2.16) holds B-equivariantly. (This was observed, (not for the first time) in [21], and has an interesting application concerning the Steinberg representation of a finite Chevalley group.) However this proof does not work in the topological case; simple counterexamples show that the extension will be discontinuous.

In many cases the Bruhat decomposition of G/P is in fact a *CW* decomposition. The following axioms are convenient in this regard:

(2.20) *Axiom.* For each $s \in S$, the projection $P_s \to P_s/B$ has a local section.

(2.21) *Axiom.* For each $s \in S$, P_s/B is homeomorphic to a sphere of positive dimension.

We then have:

(2.22) THEOREM. *Let* (G, B, N, S) *be a topological Tits system satisfying axioms 2.11, 2.20 and 2.21. Let* $P \equiv P_I$ *be a parabolic subgroup,* $I \leqslant S$, *and give* G/P *the compactly generated topology. Then*

(a) *Axiom 2.12 is satisfied.*

(b) *The Bruhat decomposition of* G/P *is a* CW *decomposition, and the closure relations on the cells are given by the Bruhat order on* W^I.

(c) *The building* $\mathscr{B}_G$ *satisfies the cofibration condition of Theorem 2.16.*

Proof. By assumption there are maps $D^{m(s)} \xrightarrow{\varphi_s} P_s/B$ such that $\varphi_s^{-1}(B) = \partial D^{m(s)}$ and $D^{m(s)}/\partial D^{m(s)} \to P_s/B$ is a homeomorphism. Furthermore φ_s lifts to a map $\tilde{\varphi}_s: D^{m(s)} \to P_s$ with $1 \in \tilde{\varphi}_s(\partial D^{m(s)})$. Thus, in Axiom (2.12) we may take $A_s = \tilde{\phi}_s(\mathring{D}^{m(s)})$, proving (a). Since P is closed (2.13c), G/P is a Hausdorff space. If $w \in W^I$ has reduced decomposition $w = s_1 \cdots s_k$, the Steinberg lemma (2.9) shows that the multiplication map $D^{m(s_1)} \times \cdots \times D^{m(s_k)} \to \bar{E}_w$ (using $\tilde{\varphi}_{s_i}$) is a characteristic map for the cell E_w. The boundary of each cell is a finite union of cells of lower dimension by 2.13a, and G/P has the weak topology by assumption. The closure relations also follow from (2.13). This proves (b). For (c) we observe that $\mathscr{B}_G$ (with the compactly generated topology) is itself a CW-complex, and the filtrations $F_k\mathscr{B}_G$ are subcomplexes: Indeed if we regard $\mathscr{B}_G$ as a quotient space of $\coprod_{I \leqslant S}(G/P_I \times \Delta_I)$, it is clear that there is one cell for each $I < S$ and $w \in W^I$. □

If G, P_I are as in the above theorem, and $w \in W^I$ has reduced decomposition $w = s_1 \cdots s_k$, let $n(w) = n(s_1) + \cdots + n(s_k)$. Thus $n(w) = \dim E_w$ and so in particular is independent of the choice of reduced decomposition. Now whenever a space has a locally finite cell decomposition, we have a *cell series* $\sum a_i t^i$, where a_i is the number of cells of dimension i. We then have:

(2.23) COROLLARY. G/P_I *admits a* CW*—decomposition with cell series* $\sum_{w \in W^I} t^{n(w)}$. □

Note also:

(2.24) COROLLARY. *If* W *is finite with maximal length element* w_0, $\mathscr{B}_G$ *is a sphere of dimension* $n(w_0) + l - 1$. □

We conclude this section with two "classical" examples. Let G be a semisimple compact Lie group and consider the Tits system (G, B, N, S), where B is a Borel subgroup, etc. First we claim that this is a topological Tits system satisfying all four of our axioms. Since W is finite, (2.11) is trivially satisfied. In (2.12) we can take A_s to be the "little $SU(2)$" (or $PSU(2)$) G_s (P_s has Iwasawa decomposition $P_s = G_s B$). In any case there is a commutative diagram

$$\begin{array}{ccc} G_s & \to & P_s \\ \downarrow & & \downarrow \\ \mathbf{C}P^1 = G_s/G_s \cap T & = & P_s/B \end{array}$$

which proves (2.20), (2.21), and hence (2.12) simultaneously. The Bruhat decomposition of $G_{\mathbf{C}}/P_I$, P_I parabolic, is then the classical Schubert cell decomposition of the flag variety $G_{\mathbf{C}}/P_I$. We have $n(s) = 2$ for all s, so $n(w) = 2l(w)$ for all $w \in W^I$. In particular the associated building $\mathscr{B}_{G_{\mathbf{C}}}$ is a sphere of dimension $2l(w_0) + l - 1$ (since $l(w)_0$ is the number of positive roots, this is exactly dim $G - 1$).

The second example (which is a generalization of the first) involves symmetric spaces G/K and the associated semisimple real Lie group $G_{\mathbf{R}}$ as in § 1. Thus $G_{\mathbf{R}}$ is the fixed group of the involution σ on $G_{\mathbf{C}}$. Now σ need not preserve the Borel subgroup B of $G_{\mathbf{C}}$, but it does preserve the parabolic Q associated to the black nodes of the Satake diagram. We will write $B_{\mathbf{R}}$, $N_{\mathbf{R}}$, $W_{\mathbf{R}}$, $S_{\mathbf{R}}$ for Q^σ, $N_K t_m$, $W_{G/K}$, $S_{G/K}$, respectively.

(2.25) THEOREM. $(G_{\mathbf{R}}, B_{\mathbf{R}}, N_{\mathbf{R}}, S_{\mathbf{R}})$ *is a topological Tits system satisfying the four axioms.* □

A proof that this is a Tits system can be found in [33]. The parabolic subgroups of $G_{\mathbf{R}}$ are related in an obvious way to those of $G_{\mathbf{C}}$: Given $I \subset S_{\mathbf{R}}$, let I' be the corresponding set in S (see § 1). We denote by $\mathcal{O}_I$ the parabolic in $G_{\mathbf{R}}$ generated by $B_{\mathbf{R}}$ and I. Then $\mathcal{O}_I = (P_{I'})^\sigma$. ($B_{\mathbf{R}}$ is usually called a "minimal parabolic", but this terminology conflicts with our use of the term. From the point of view of Tits systems, it is precisely analogous to the Borel subgroup of $G_{\mathbf{C}}$—although in general it is neither solvable nor connected.) The rest of the theorem is also easily deduced from [33]; the details will be omitted, but see § 5. The main point is that for the minimal parabolics $\mathcal{O}_i$, $\mathcal{O}_i/B_{\mathbf{R}}$ is a sphere of dimension n_i.

As for the building, one can deduce from (2.24) that it is a sphere whose dimension is dim $G/K - 1$. However it is an interesting fact, that does not seem to appear in the literature, that the building can be canonically identified with the "tangent cut locus" of G/K: first recall (cf. [10], [20]) that if M is a compact Riemannian manifold and p is a fixed point of M, a point

x is a *cut point* (with respect to p) if there is a geodesic from p to x that minimizes arc length up to x but no further. The *cut locus* is the set of cut points. Similarly a vector X in the tangent space T_p is a tangent *cut point* if $\exp_p X$ is a cut point along the geodesic $\exp_p(tX)$. The *tangent cut locus* is the set of all such points in T_p, and is homeomorphic to the unit sphere in T_p. When $M = G/K$ we take $p = 1$.

(2.26) THEOREM. *Let G/K be a simply-connected symmetric space, with G simple. Then the tangent cut locus is precisely the K-orbit in $\mathfrak{m}$ of the outer wall of the Cartan simplex Δ_m. It is therefore canonically identified with the topological building of the associated real form $G_{\mathbf{R}}$.*

As usual, the assumption G simple is just for convenience. We sketch the proof: the first assertion is a fairly easy consequence of Theorem (1.8), and is left to the reader. Now consider the building $\mathscr{B}_{G_{\mathbf{R}}}$. It is a quotient space of $G_{\mathbf{R}}/B_{\mathbf{R}} \times \Delta_0 = K/C_K t_m \times \Delta_0$, where Δ_0 is a simplex of dimension (rank G/K)-1; we take Δ_0 to be the outer wall of $\Delta_{\mathfrak{m}}$. For each $I \leqslant S_{G/K}$, let Δ_I temporarily denote the corresponding face of Δ_0; *i.e.* $\{X \in \Delta_0 : \alpha_i(x) = 0 \; \forall \, i \in I\}$. Then the K-orbit of Δ_0 in $\mathfrak{m}$, $K\Delta_0$, is also a quotient of $K/C_K t_{\mathfrak{m}} \times \Delta_0$. The relations are $(k_1 X) \sim (k_2 X)$ if $X \in \mathring{\Delta}_I$ and $k_1 = k_2 \bmod K_I$. But $K_I = K \cap \mathcal{O}_I$, so these relations are identical to the ones that define the building. □

§ 3. LOOP GROUPS

Let LG, $LG_{\mathbf{C}}$ denote the free loop spaces. Let $G_{\mathbf{C}}$ denote the group of loops which are restrictions of regular maps $\mathbf{C}^* \to G_{\mathbf{C}}$, and let $L_{alg}G = L_{alg}G_{\mathbf{C}} \cap LG$. Thus if we fix an embedding $G_{\mathbf{C}} \subset GL(n, \mathbf{C})$, $L_{alg}G$ consists of the loops f in LG admitting a finite Laurent expansion $f(z) = \sum_{k=-m}^{m} A_k z^k$, whereas $L_{alg}G_{\mathbf{C}}$ consists of the loops f in $LG_{\mathbf{C}}$ such that both f and f^{-1} admit finite Laurent expansions. We will also write $\tilde{G}_{\mathbf{C}}$ for $L_{alg}G_{\mathbf{C}}$. In fact $\tilde{G}_{\mathbf{C}}$ is the group of points over $\mathbf{C}[z, z^{-1}]$ of the algebraic group $G_{\mathbf{C}}$. Its Lie algebra is the loop algebra $\tilde{g}_{\mathbf{C}}$ of regular maps $\mathbf{C}^* \to g_{\mathbf{C}}$. The integer m in the above Laurent expansion defines a filtration of $\tilde{G}_{\mathbf{C}}$ by finite dimensional subspaces; we give $\tilde{G}_{\mathbf{C}}$ the corresponding weak topology.

Let P denote the subgroup of $\tilde{G}_{\mathbf{C}}$ consisting of regular maps $\mathbf{C} \to G_{\mathbf{C}}$ (i.e. maps with nonnegative Laurent expansion, or $G_{\mathbf{C}[z]}$), and let $\tilde{B}$ denote the Iwahori subgroup: $\{f \in P : f(0) \in B^-\}$. Finally, let $\tilde{N} = L_{alg}N_{\mathbf{C}}$, and recall that $\tilde{W}$ can be regarded as a "subgroup" of $\tilde{G}_{\mathbf{C}}$, since $R \leqslant \mathrm{Hom}(S^1, T) \leqslant L_{alg}T$. More precisely, we have $\tilde{N}/T_{\mathbf{C}} = \hat{W}$, and $\tilde{W} \subset \hat{W}$.

The *affine root system* $\tilde{\Phi}$ is the set $\mathbf{Z} \times \Phi$. It can be thought of as a set of affine linear functionals on t, but for our purposes it is just a device for encoding combinatorial information about the affine Weyl group and $\tilde{G}_{\mathbf{C}}$. In particular, to each $(n, \alpha) \in \tilde{\Phi}$ we associate a root subalgebra $X_{n,\alpha}$ of $\tilde{g}_{\mathbf{C}}$ consisting of the regular maps $\mathbf{C}^* \to X_\alpha$ homogeneous of degree n. These subalgebras are one—dimensional, and are precisely the nontrivial eigenspaces of the following T^{l+1} action: The constant loops T^l act in the obvious way, and the extra S^1 factor acts by rotating the loops. We also have root subgroups $U_{(n,\alpha)} = \exp X_{n,\alpha} \leqslant \tilde{G}_{\mathbf{C}}$. One can easily check that $\tilde{W}$ (acting by left conjugation) permutes the root subgroups. The resulting action of $\tilde{W}$ on $\tilde{\Phi}$ is given by $(w\lambda) \cdot (n, \alpha) = (n+\alpha(\lambda), w\alpha)$ for $\lambda \in \hom(S^1, T)$, $w \in W$. The various additional structures associated with ordinary root systems can be defined here as well. The positive roots $\tilde{\Phi}^+$ are the (n, α) with $n \geqslant 1$ or $n = 0$ and $\alpha < 0$ (note these correspond to the Iwahori subgroup $\tilde{B}$); the remaining roots are negative. As in the finite case, the length of an element σ in $\tilde{W}$ is equal to the number of positive roots taken to negative roots by σ (in particular this latter number is finite, as is clear anyway from the above formula for the $\tilde{W}$ action). The simple affine roots are defined as the set of elements of $\tilde{\Phi}^+$ which are indecomposable with respect to addition: $(m, \alpha) + (n, \beta) = (m+n, \alpha+\beta)$ (if $\alpha+\beta$ is a root). Hence the simple roots are $(0, -\alpha), \cdots (0, -\alpha_l)$ and $(1, \alpha_0)$.

To each root (n, α), we can also associate a "little SL_2" subgroup generated by $U_{n,\alpha}$ and $U_{-n,-\alpha}$. In particular $\tilde{G}_{\mathbf{C},i}$ is the subgroup corresponding to the ith simple affine root, $0 \leqslant i \leqslant l$. Thus $\tilde{G}_{\mathbf{C},i} = G_{\mathbf{C},i}$ if $i \neq 0$, and $\tilde{G}_{\mathbf{C},0}$ corresponds to $(1, \alpha_0)$. For example, if $G = SU(2)$, $\tilde{G}_{\mathbf{C},0}$ is the subgroup of matrices $\begin{pmatrix} a & bz \\ cz^{-1} & d \end{pmatrix}$ with $ad - bc = 1$. We let $\tilde{G}_i = \tilde{G}_{\mathbf{C},i} \cap LG$. Again $\tilde{G}_i = G_i$ if $i \neq 0$. Note that for all i, evaluation at $z = 1$ gives an isomorphism $\tilde{G}_i \xrightarrow{\cong} G_i \cong SU(2)$.

(3.1) THEOREM. *Assume G is simply-connected. Then $(\tilde{G}_{\mathbf{C}}, \tilde{B}, \tilde{N}, \tilde{S})$ is a topological Tits system satisfying the four axioms of § 2.*

Proof. That $(\tilde{G}_{\mathbf{C}}, \tilde{B}, \tilde{N}, \tilde{S})$ is a Tits system in the ordinary sense is essentially due to Iwahori and Matsumoto [16]. (They work over a complete local field K; here we take K to be the field of infinite Laurent series bounded below. It is not hard to get from the Chevalley group G_K to $G_{\mathbf{C}[z,z^{-1}]} = \tilde{G}_{\mathbf{C}}$.) See also Kac and Peterson [17].

Clearly $\tilde{B}$ and $\tilde{N}$ are closed subgroups and $\tilde{W}$ is discrete. For Axiom (2.11) we need to show that if $\tilde{W}$ is an irreducible affine Weyl group, and I is a proper subset of $\tilde{S}$, then $\tilde{W}_I$ is finite. This is obvious since the elements of I have a common fixed point (i.e. the intersection of the corresponding reflection hyperplanes is nonempty). In Axiom (2.12) we take $A_s = \tilde{G}_s$. We have $\tilde{G}_s\tilde{B} = \tilde{G}_{\mathbf{C},s}\tilde{B} = \tilde{B} \quad U_s s\tilde{B} = P_s$. In particular $P_s/\tilde{B} = \tilde{G}_s/(\tilde{G}_s \cap \tilde{B}) \cong SU(2)/T = \mathbf{C}P^1$, which also proves Axioms (2.20) and (2.21). □

(3.2) COROLLARY. *$\Omega_{alg}G$ is a CW-complex with cells of even dimension, indexed by $\mathrm{Hom}(S^1, T)$. The Poincaré series for its integral homology is $\sum_{\lambda \in \mathrm{Hom}(S^1,T)} t^{2\bar{l}(\lambda)}$, where $\bar{l}(\lambda)$ is the minimal length accuring in λW. Identifying $\mathrm{Hom}(S^1, T)$ with $\tilde{W}^S$, the closure relations on the cells are given by the Bruhat order on $\tilde{W}^S$.* □

Remark. An explicit formula for $\bar{l}(\lambda)$ is given in [16], Prop. 1.25: $\bar{l}(\lambda) = (\sum_{\alpha > 0} |\alpha(\lambda)|) - |\{\alpha > 0 : \alpha(\lambda) > 0\}|$.

We will also need the "Iwasawa decomposition" (see [17], [27], [29]):

(3.3) THEOREM. $\tilde{G}_{\mathbf{C}} = \Omega_{alg}G \times P$. □

Remark. Note that (3.3) shows that the associated building, which we will be denoted simply by $\mathscr{B}_G$, is a quotient of $L_{alg}G/T \times \Delta$. The equivalence relation is then $(f_1T, X) \sim (f_2T, X)$ if $X \in \mathring{\Delta}_I$ and $f_1 = f_2 \bmod LG \cap P_I$.

§ 4. QUILLEN'S THEOREM FOR LOOP GROUPS

In this section we will give Quillen's proof of the following theorem.

(4.1) THEOREM. *Let G be a compact Lie group. Then the inclusion $\Omega_{alg}G \to \Omega G$ is a homotopy equivalence.*

If G is simply connected, let $\mathscr{B}_G$ denote the topological building associated to the algebraic loop group $L_{alg}G_{\mathbf{C}}$ as in § 2.

(4.2) THEOREM (Quillen). *$\Omega_{alg}G$ acts freely on $\mathscr{B}_G$, with orbit space G.*

Proof of (4.1). It is easy to reduce to the case when G is simply connected. Since B_G is contractible by Theorem 2.16, we conclude at once from Theorem (4.2) that $\Omega_{alg}G \to \Omega G$ is a weak equivalence. Since both spaces have the homotopy type of a CW-complex, the map is in fact a homotopy equivalence. □

Since G is a product of simple groups (as is $G_{\mathbf{C}}$), it is very easy to reduce to the case when G is simple. For the rest of this section, then, we assume G is simple and simply-connected, of rank l.

To prove 4.2, we introduce Quillen's space of special paths $\mathscr{S}_G$: this is the space of all paths $[0, 1] \to G$ of the form $f(e^{2\pi it}) \exp tX$, where $f \in \Omega_{alg}G$ and $X \in \mathfrak{g}$. $\mathscr{S}_G$ is topologized as a quotient of $\Omega_{alg}G \times \mathfrak{g}$. Note that $L_{alg}G$ acts on $\mathscr{S}_G$ by $h \cdot (f \exp tX) = hf \exp tXh(1)^{-1}$. The following key lemma, whose proof is deferred, also helps to explain the significance of the parabolic subgroups P_I.

(4.3) LEMMA. *Suppose* $X \in \mathring{\Delta}_I$, *then the isotropy group of* $\exp tX$ *is* $L_{alg}G \cap P_I$.

(4.4) THEOREM (Quillen). $\mathscr{S}_G$ *is* $L_{alg}G$*-equivariantly homeomorphic to the building* $\mathscr{B}_G$.

Proof. The action map $\varphi: L_{alg}G \times \Delta \to \mathscr{S}_G$ given by

$$\varphi(f, X) = f \exp tXf(1)^{-1}$$

is surjective by Theorem 1.1. If $\varphi(f_1, X_1) = \varphi(f_2, X_2)$, then (evaluating at $t=1$) $\exp X_1$ and $\exp X_2$ are conjugate in G, so $X_1 = X_2$ by Theorem 1.3. We then have $\varphi(f_1, X) = \varphi(f_2, X)$ if and only if $f_1 = f_2$ mod the isotropy group of $\exp tX$. Hence, by (4.3), φ factors through the desired homeomorphism $\mathscr{B}_G \to \mathscr{S}_G$. □

Remark. Here we have used the Iwasawa decomposition (3.3) to identify $\mathscr{B}_G = (\tilde{G}_{\mathbf{C}}/\tilde{B} \times \Delta)/\sim$ with $(L_{alg}G/T \times \Delta)/\sim$.

(4.5) LEMMA. $L_{alg}G \cap P_I$ *is generated by* T *and the subgroups* $\tilde{G}_i, i \in I$.

Proof. We have $P_I = \tilde{B}W_I\tilde{B}$. By the Steinberg lemma (2.9), each $\tilde{B}w\tilde{B}(w \in W_I)$ has the form XB, where X is a product of the $\tilde{G}_i$. Since $L_{alg}G \cap XB = XT$, the lemma follows. □

Proof of 4.2. The action of $\Omega_{alg}G$ on $\mathscr{S}_G$ is clearly free. By (4.4), the same is true for $\mathscr{B}_G$. Now consider the orbit space $\mathscr{B}_G/\Omega_{alg}G$. Since $\mathscr{B}_G = (L_{alg}G/T \times \Delta)/\sim\ = (\Omega_{alg}G \times G/T \times \Delta)/\sim$, the orbit space is a quotient of $G/T \times \Delta$. The equivalence relation is given by $(g_1T, X) \sim (g_2T, X)$ if $X \in \mathring{\Delta}_I$ and $g_2 = fg_1p$ with $f \in \Omega_{alg}G$, $p \in P_I$. In fact $p \in LG \cap P_I$. Now let $\bar{G}_I = e(LG \cap P_I)$, where e is evaluation at $z = 1$. Then $(g_1T, X) \sim (g_2T, X)$ if and only if $g_1 = g_2 \bmod \bar{G}_I$. For if $g_2 = f\,g_1\,p$ as above, then $\bar{G}_I = e(L_{alg}G \cap P_I)$, where e is evaluation at $z = 1$. Then $(g_1T, X) \sim (g_2T, X)$ if and only if $g_1 = g_2 \bmod \bar{G}_I$. For if $g_2 = f\,g_1\,p$ as above, then

$g_2 = f\, g_1\, p(1)$, and conversely if $g_2 = g_1\, p(1)$, then $g_2 = f\, g_1\, p$, where $f = g_2\, p^{-1}\, g_1^{-1}$. But by (4.5), $\bar{G}_I = G_I$ (see § 1). In other words, the equivalence relation here coincides with the classical relation of Theorem 1.5, which has quotient G. □

Proof of 4.3. Fix $X \in \mathring{\Delta}_I$. We first show that $L_{alg}G \cap P_I$ fixes $\exp tX$ in $\mathscr{S}_G$. By (4.5) it is enough to show that each $\tilde{G}_i (i \in I)$ fixes

$$\exp tX : f(e^{2\pi i t}) \exp tX f(1)^{-1} = \exp tX .$$

If $i \neq 0$, $\tilde{G}_i = G_i$ is a subgroup of the constant loops, so f is a constant $g \in G_i$. The desired equation is then equivalent to $g \cdot X = X$ (recall that $g \cdot X = Ad(g)X$). But since $i \neq 0$, $\alpha_i(X) = 0$, so this is true by definition. Now suppose $i = 0$, so that X lies on the outer wall: $\alpha_0(X) = 1$. Then $X = \frac{1}{2}\alpha_0^* + Y$, where $\alpha_0^* = 2\alpha_0/\alpha_0 \cdot \alpha_0$ is the coroot of α_0 and $\alpha_0(Y) = 0$.

The equation we want can be written $(f \in \tilde{G}_0)$:

$$f(e^{2\pi i t}) = \exp tX\ f(1) \exp -tX$$

Since $f(1) \in G_0$, $f(1) \cdot Y = Y$, and our equation simplifies to

$$f(e^{2\pi i t}) = \exp\left(\frac{1}{2} t\alpha_0^*\right) f(1) \exp\left(-\frac{1}{2} t\alpha_0^*\right)$$

Note this is now an equation in the path space of G_0. Identifying G_0 with $SU(2)$, it can be written

$$\begin{pmatrix} a & be^{2\pi i t} \\ ce^{-2\pi i t} & d \end{pmatrix} = \begin{pmatrix} e^{\pi i t} & 0 \\ o & e^{-\pi i t} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{-\pi i t} & o \\ o & e^{\pi i t} \end{pmatrix}$$

Where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$. This last equation is obviously correct, and we conclude that $L_{alg}G \cap P_I$ fixes $\exp tX$.

Conversely, suppose

$$f \exp tX f(1)^{-1} = \exp tX , \quad \text{or} \quad f = \exp tX\ f(1) \exp(-tX) .$$

Then $f(1) \in C_G \exp X = G_I$, and hence $f(1) = h(1)$ for some $h \in L_{alg}G \cap P_I$. But then $h = \exp tXh(1) \exp -tX = f$.

A useful fact that follows from all this is:

(4.6) THEOREM. *Evaluation at 1 induces an isomorphism* $L_{alg}G \cap P_I \cong G_I$. *In particular,* $L_{alg}G \cap P_I$ *is a compact Lie group.*

Proof. We have seen that e maps $L_{alg}G \cap P_I$ onto G_I. The kernel is $\Omega_{alg}G \cap P_I$. But $\Omega_{alg}G$ acts freely on $\mathcal{S}_G$, and $L_{alg}G \cap P_I$ fixes Δ_I, so $\Omega_{alg}G \cap P_I = \{1\}$.

Remark. As always, I is a proper subset of $\tilde{S}$ in (4.6). Of course (4.6) also depends on our assumption that G is simple. Its discrete analogue is the fact that W_I is finite if $\tilde{W}$ is irreducible. (It may be helpful to consider the "discrete" versions of all the results of this section. For example, the discrete version of "$\Omega_{alg}G$ acts freely on B_G" is "the coroot lattice $\operatorname{Hom}(S^1, T)$ acts freely on t (the foundation of $\mathcal{B}_G$)"; of course the latter assertion is trivial).

Note that we have shown that $\mathcal{S}_G/\Omega_{alg}G = G$, and in fact the orbit map $\mathcal{S}_G \to G$ is given by evaluation at $t = 1$. This can also be proved directly. It reduces to the following interesting theorem, also observed by Quillen.

(4.7) THEOREM. *Suppose* $X, Y \in \mathfrak{g}$ *and* $\exp X = \exp Y$. *Then* $\exp tX = f(e^{2\pi i t}) \exp tY$ *for some* $f \in \Omega_{alg}G$. □

It is not hard to prove this directly—for example, it is enough to prove it for $G = U(n)$. Not surprisingly, however, it is also implicit in what we have already one. First, one can reduce to the case when G is simple and simply-connected. Using (1.3), one can easily reduce further to the case $X \in \mathring{\Delta}_I$, $Y = g \cdot X$ for some $g \in G$. Then $g \in C_G \exp X = G_I$, so $g = h(1)$ with $h \in L_{alg}G \cap P_I$. Let $h = \exp tX\, g \exp -tX$; then $h \in L_{alg}G$ and $f = hh(1)^{-1}$ is the desired loop.

§ 5. THE LOOPS ON A SYMMETRIC SPACE

We assume throughout this section that G is simple and simply—connected. If σ is an involution on G with fixed group K, as usual, then K is connected and G/K is simply—connected. The notations and conventions of § 1 and § 3 remain in force.

The loop space $\Omega(G/K)$ is homotopy equivalent to the space of paths in G that start at the identity and end in K. Now consider the involution τ on ΩG given by $\tau(f)(z) = \sigma(f(\bar{z}))$. The fixed group $(\Omega G)^\tau$ is clearly homeomorphic to our space of paths, since $f \in (\Omega G)^\tau$ implies $f(-1) \in K$.

Henceforth we will always consider $(\Omega G)^\tau$ in place of $\Omega(G/K)$. Note also the definition of τ extends to LG, $LG_{\mathbf{C}}$, and even $L_{alg}G_{\mathbf{C}}$: for if $f: \mathbf{C}^* \to G_{\mathbf{C}}$ is a regular map, so is $\sigma \circ f \circ (z \mapsto \bar{z})$, since σ is anti—complex on $G_{\mathbf{C}}$.

(5.1) THEOREM (Quillen). *The inclusion* $(\Omega_{alg}G)^\tau \to (\Omega G)^\tau$ *is a homotopy equivalence.*

We defer the proof to the end of this section.

Thus $\Omega(G/K)$ can be thought of as a real form of $\Omega_{alg}G_{\mathbf{C}}$. More precisely, $(L_{alg}G_{\mathbf{C}})^\tau$ is a real form of $L_{alg}G_{\mathbf{C}}$, and $\Omega(G/K)$ is a homogeneous space of this real form. For clearly P (regular maps $\mathbf{C} \to G_{\mathbf{C}}$) is invariant under τ, so from (3.3) we obtain a corresponding "Iwasawa" decomposition.

(5.2) THEOREM. *The multiplication map* $(\Omega_{alg}G)^\tau \times P^\tau \to (L_{alg}G_{\mathbf{C}})^\tau$ *is an homeomorphism.* □

On the other hand $\tilde{B}$ is of course not τ-invariant in general, since B is not σ-invariant. However the parabolic subgroup $\tilde{Q}$ corresponding to the black nodes on the extended Satake diagram is clearly τ-invariant; in fact $\tilde{Q} = Q \times U^\#$, where $U^\# = \{f \in P: f(0) = 1\}$ (note $U^\#$ is τ-invariant). Now consider $\tilde{N}_{\mathbf{C}} = L_{alg}N_{\mathbf{C}}$. Since σ preserves $N_{\mathbf{C}}$, τ preserves $\tilde{N}_{\mathbf{C}}$. Note $\mathrm{Hom}(S^1, T)$ is also τ invariant and in fact if $f \in \mathrm{hom}(S^1, T)$, $\tau f = \sigma(f(z)^{-1})$. It follows that $(\mathrm{hom}(S^1, T))^\tau = \mathrm{hom}(S^1, T_{\mathfrak{m}}) \cong R_{\mathfrak{m}}$. It is also easy to see that $\tilde{N}_{\mathbf{C}}^\tau \cap \tilde{Q}$ is normal in $(\tilde{N}_{\mathbf{C}})^\tau$; the quotient is $\tilde{W}_{\mathbf{R}}$. Here we recall that $\tilde{W}_{\mathbf{R}}$ is the affine Weyl group associated to the restricted root system Σ; it has a canonical set of Coxeter generators $\tilde{S}_{\mathbf{R}}$. Write $\tilde{G}_{\mathbf{R}}$, $\tilde{B}_{\mathbf{R}}$, $\tilde{N}_{\mathbf{R}}$, for $(\tilde{G}_{\mathbf{C}})^\tau$, $\tilde{Q}^\tau$, $\tilde{N}_{\mathbf{C}}^\tau$, respectively.

(5.3) THEOREM. $(\tilde{G}_{\mathbf{R}}, \tilde{B}_{\mathbf{R}}, \tilde{N}_{\mathbf{R}}, \tilde{S}_{\mathbf{R}})$ *is a topological Tits system satisfying the four axioms (2.11), (2.12), (2.20) and (2.21).*

Before giving the proof, we discuss some corollaries. If $I \subset \tilde{S}_{\mathbf{R}}$, we let $\tilde{Q}_I$ denote the parabolic subgroup $P_{I'}$ of $\tilde{G}_{\mathbf{C}}$; here I' consists of the black nodes of the extend ended Satake diagram together with the white nodes that correspond under restriction to elements of I (for example, $\tilde{Q} = \tilde{Q}_\phi$). Then $\tilde{Q}_I$ is τ-invariant and the parabolic subgroups (containing $\tilde{Q}^\tau$) are precisely the subgroups $\tilde{Q}_I^\tau$. Let $\mathcal{O}_I = \tilde{Q}_I^\tau$. The proof of (5.3) will show that for the minimal parabolics $\mathcal{O}_s$, $s \in \tilde{S}_{\mathbf{R}}$, $\mathcal{O}_s/\tilde{B}_{\mathbf{R}}$ is sphere of dimension $n(s) \equiv m(\alpha_s) + m(2\alpha_s)$ (here the multiplicity $m(2\alpha_s)$ is of course zero if $2\alpha_s$ is not a root). If $s_1 \ldots s_k$ is a reduced decomposition of $w \in W^I$, let $n(w) = n(s_1) + \cdots + n(s_k)$.

(5.4) COROLLARY. *The Bruhat decomposition of* $\tilde{G}_{\mathbf{R}}/\mathcal{O}_I$ *is a* CW *decomposition, and the closure relations on the cells are given by the Bruhat order on* $\tilde{W}^I_{\mathbf{R}}$. *Furthermore the cell series is* $\sum_{w \in \tilde{W}^I_{\mathbf{R}}} t^{n(w)}$. □

(5.5) COROLLARY (Bott-Samelson). $\Omega G/K$ *has the homotopy type of a CW-complex with cell series* $\sum_{w \in \tilde{W}^I_{\mathbf{R}}} t^{n(w)}$, *where* $I = S_{\mathbf{R}}$. □

The cell series obtained by Bott and Samelson ([7], Corollary 3.10) is described in terms of the diagram for $\mathfrak{t}_{\mathfrak{m}}$, but can be shown to agree with the one above (cf. [25] for the case of ΩG). Bott and Samelson also showed that the cells they constructed are all cycles mod 2. Here, reverting temporarily to the notation of § 2, their result appears in the following form.

(5.6) THEOREM. *Let* (G, B, N, S) *be a topological Tits system satisfying the four axioms, and let* P *be a parabolic subgroup. Then the Bruhat cells of* G/P *are all cycles mod 2.*

Proof. Let $P = P_I$, $I \leqslant S$, and fix $w \in W^I$. Let $s_1 \dots s_k$ be a reduced decomposition of w. If $k = 1$ then P_{s_1}/B is a sphere and maps homeomorphically onto $\bar{E}_{s_1}$ by $xB \mapsto xP$. Hence E_w is an integral cycle. In general, consider the space $X_w = P_{s_1} \times_B P_{s_2} \times_B \dots \times_B P_{s_k}/B$, and let $w' = s_2 \dots s_k$. By assumption each projection $P_s \to P_s/B$ is a locally trivial principal B-bundle, so the natural projection $X_w \to P_{s_1}/B$ is a locally trivial fibre bundle with fibre $X_{w'}$. Hence we conclude by induction on k that X_w is a topological manifold (not necessarily orientable). The fundamental class in mod 2 homology is represented by the cell $A_{s_1} \times A_{s_2} \dots \times A_{s_k}$ in X_w, where $A_s \leqslant P_s$ is chosen as in the proof of theorem 2.22, and by the Steinberg lemma (2.9) this cell is carried homeomorphically onto E_w under the natural (multiplication) map $X_w \to G/P$. This proves the theorem. □

Returning to our standard notation, we have:

(5.7) COROLLARY (Bott-Samelson). $\Omega G/K$ *has* mod 2 *Poincaré series as in (5.5).* □

In general one could ask for a combinatorial formula describing the differential in the cellular chain complex: $\partial[E_w] = \sum_{x \to w} a_x[E_x]$, where the sum is over the $x \in W^I$ that immediately precede w in the Bruhat order, and satisfy $n(x) + 1 = n(w)$. The problem is to determine the integers a_x.

Of course if the multiplicities $m(\alpha_s)$, $m(\alpha_{2s})$ are all even, every cell is an integral cycle. Here we recall that the multiplicities are all even if and only if G/K is of "splitting rank" (not to be confused with the split form mentioned earlier): that is, rank K + rank G/K = rank G. For example, G itself, regarded as a symmetric space, is of splitting rank, as is $SU(2n)/Sp(n)$.

(5.8) COROLLARY. *If G/K is of splitting rank, the integral homology of $\Omega G/K$ is concentrated in even dimensions, and the Poincaré series is given by the series of (5.5).*

The "somewhat mysterious application..." of Bott-Samelson ([7], 4.1) is quite transparent from the present point of view.

(5.9) THEOREM (Bott-Samelson). *Suppose* rank G/K = rank G *(i.e., $G_{\mathbf{R}}$ is the split real form of $G_{\mathbf{C}}$). Then* $\dim H_q(\Omega G/K, \mathbf{Z}/2) = \dim H_{2q}(\Omega G; \mathbf{Z}/2)$. *Hence the* mod 2 *Poincaré series of $\Omega G/K$ is $\prod_{i=1}^{l}(1-t^{m_i})^{-1}$, where the m_i are the exponents of G.*

Proof. By assumption, $\mathfrak{t}_{\mathfrak{m}} = \mathfrak{t}$. It follows at once that τ preserves $\tilde{B}$ and is the identity on $\tilde{W}$; hence τ preserves the Bruhat cells in $\tilde{G}_{\mathbf{C}}/P$. Furthermore, each cell is identified with a complex vector space in such a way that τ corresponds to a linear conjugation. Since every cell is a cycle mod 2, this proves the theorem. (In more detail, σ preserves the root subalgebras X_α, and of course acts anti-linearly. The same is true for τ acting on the $X_{n,\alpha}$, and hence (by definition) for τ acting on the root subgroups exp $X_{n,\alpha}$. In particular τ acts by a conjugation on each U_s, $s \in \tilde{S}$. But every cell can be identified with a product of subgroups U_s, by the Steinberg lemma.) □

Remark. Bott and Samelson obtain similar results with $\Omega(G/K)$ replaced by suitable homogeneous spaces of K. For example, if rank G/K = rank G, they show that $\dim H_q(K/C_k t_m; \mathbf{Z}/2) = \dim H_{2q}(G/T, \mathbf{Z}/2)$. These results also fit neatly into the present context, using the topological Tits system $(G_{\mathbf{R}}, B_{\mathbf{R}}, N_{\mathbf{R}}, S_{\mathbf{R}})$. The points is that $G/T = G_{\mathbf{C}}/B$, $K/C_k t_m = G_{\mathbf{R}}/B_{\mathbf{R}}$, etc.

Proof of Theorem 5.3. Axiom (2.1) is easy and is left to the reader. The proof of the remaining three axioms for an ordinary Tits system follows a standard pattern and will only be sketched. The first step is to prove the Bruhat decomposition directly. One way of doing this, which is of some independent interest, is sketched in § 8. Briefly, the argument is as follows. The $\tilde{Q}$-orbits in $\tilde{G}_{\mathbf{C}}/\tilde{Q}$ are vector bundles over certain flag varieties, and

τ acts on each orbit as a conjugate linear bundle automorphism. For the orbit $\tilde{Q}w\tilde{Q}/\tilde{Q}$, this action is free on the base unless $w \in \tilde{W}_{\mathbf{R}}$. Furthermore, if $w \in \tilde{W}_{\mathbf{R}}$ then $\tilde{Q}\tilde{w}\tilde{Q} = \tilde{B}w\tilde{Q}$ so the Bruhat cell $\tilde{B}w\tilde{Q}/\tilde{Q}$ is τ-invariant. The Bruhat decomposition for $\tilde{G}_{\mathbf{R}}$ then follows by taking τ fixed points of the $\tilde{Q} - \tilde{Q}$ double coset decomposition of $\tilde{G}_{\mathbf{C}}$. In particular this proves that $\tilde{B}_{\mathbf{R}}$, $\tilde{N}_{\mathbf{R}}$ generate $\tilde{G}_{\mathbf{R}}$. Axiom (2.3) is easy. For (2.4), we use induction on $l(w)$. The inductive step reduces to showing that $s\tilde{B}_{\mathbf{R}}\, s \subseteq \tilde{B}_{\mathbf{R}} \cup \tilde{B}_{\mathbf{R}}\, s\tilde{B}_{\mathbf{R}}$, which in turn can be deduced from the Bruhat decomposition for rank one groups (already proved). (Cf. [33], Prop. 1.2.3.17, for the details of one version of this argument.)

Axiom (2.11) is immediate since $\tilde{W}_{\mathbf{R}}$ is an irreducible affine Weyl group (see § 3). For the remaining axioms, we need to explicitly construct certain subgroups $\tilde{K}_i$ (analogues of the "little $SU(2)$" subgroups in the loop group case), where $\tilde{K}_i$ corresponds to the ith simple root β_i of the affine restricted root system $\tilde{\Sigma}$. When $i \neq 0$, $\tilde{K}_i$ is the group of constant loops $K_{\beta i}$ already constructed in § 1. $\tilde{K}_0$ is constructed in the same way. Let $I \subseteq \tilde{S}$ be the subset formed by taking the union of the black nodes and the special node $-\alpha_0$ of the extended Satake diagram, and then taking the path component of $-\alpha_0$ in this smaller diagram. Let $\tilde{G}_I = L_{alg}G \cap P_I$ (compare § 4). Then $\tilde{G}_I$ and its commutator subgroup $\tilde{G}'_I$ are τ-invariant subgroups and we define $\tilde{K}_0 = (\tilde{G}'_I)^\tau$. Note that $\tilde{K}_0$ is a compact subgroup of $\tilde{G}_{\mathbf{R}}$; in fact evaluation at 1 yields an embedding $\tilde{K}_0 \to K$. (Note however that $\tilde{K}_0$ does not consist of K-valued loops.) The complexification of $\tilde{G}'_I$ is the subgroup $\tilde{G}'_{\mathbf{C},I}$ generated by the root subgroups U_i, $i \in I$. Passing to τ-fixed points we obtain a semisimple real form $\tilde{G}_{\mathbf{R},0}$ with $\tilde{K}_0$ as maximal compact. The structure of these groups is easily read off from the Satake diagram.

Example. Let $G = SU(4)$, $K = Sp(2)$, as in § 1. Then $\tilde{S} \cong (0, 1, 2, 3)$ and $I = (0, 1, 3)$. The parabolic P_I consists of all matrices $\begin{pmatrix} A & Bz \\ Cz^{-1} & D \end{pmatrix}$ in $\tilde{G}_{\mathbf{C}}$ with A, B, C, D 2×2 matrices over $\mathbf{C}[z]$. $\tilde{G}_{\mathbf{C},I}$ consists of the elements of P_I with A, B, C, D constant; note evaluation at one is in this case an isomorphism onto the constant loops. In this example $\tilde{G}_I = \tilde{G}'_I \cong SU(4)$ and $\tilde{K}_0$ is the subgroup of matrices as above with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2)$. In particular $\tilde{K}_0$ is isomorphic to $Sp(2)$; note this in fact follows immediately from the Satake diagram.

Now let $\mathcal{O}_i$ be the minimal parabolic $\langle \tilde{B}_{\mathbf{R}}, s_i \rangle \leqslant \tilde{G}_{\mathbf{R}}$, as usual. In Axiom (2.12) we take $A_i = \tilde{K}_i$. Certainly $\tilde{K}_i$ is compact and contains 1, and since

$\tilde{K}_i \cap \tilde{B}_{\mathbf{R}}$ is a subgroup of lower dimension, we have $\tilde{K}_i = \overline{\tilde{K}_i \cap \tilde{B}_{\mathbf{R}}\, s_i \tilde{B}_{\mathbf{R}}}$. The Iwasawa decomposition of $\tilde{G}_{\mathbf{R},i}$ shows that $\tilde{K}_i\tilde{B}_{\mathbf{R}} = \tilde{G}_{\mathbf{R},i}\tilde{B}_{\mathbf{R}}$. Now $\mathcal{O}_i = \mathbf{B}_{\mathbf{R}} \quad \mathbf{B}_{\mathbf{R}}\, s_i \mathbf{B}_{\mathbf{R}}$, and $B_{\mathbf{R}}\, s_i \mathbf{B}_{\mathbf{R}} = U_{\mathbf{R},i} s_i B_{\mathbf{R}}$, where $U_{\mathbf{R},i}$ corresponds to the positive roots β_i and (if $2\beta_i$ is a root) $2\beta_i$. Since $U_{\mathbf{R},i} \leqslant \tilde{G}_{\mathbf{R},i}$, this completes the proof of (2.12). Note $\mathcal{O}_i/\tilde{B}_{\mathbf{R}} = \tilde{K}_i/\tilde{K}_i \cap \tilde{B}_{\mathbf{R}}$. Since $U_{\mathbf{R},i}$ is homeomorphic to a real vector space of dimension $n_i = m_{\beta i} + m_{2\beta i}$, and $\mathcal{O}_i/\tilde{B}_{\mathbf{R}}$ is compact, we also conclude that $\mathcal{O}_i/\tilde{B}_{\mathbf{R}}$ is a sphere of dimension n_i, and that $\mathcal{O}_i \to \mathcal{O}_i/\tilde{B}_{\mathbf{R}}$ has a local section. This completes the proof of Theorem 5.3. □

Now let $\mathscr{B}_{G/K}$ be the building associated to the topological Tits system of (5.3). To prove Theorem 5.1, it is enough to show (as in § 4):

(5.4) THEOREM (Quillen). *$(\Omega_{alg}G)^\tau$ acts freely on $\mathscr{B}_{G/K}$, with orbit space G/K.*

Proof. $B_{G/K}$ is a quotient space of $(\Omega_{alg}G)^\tau \times K/C_K t_m \times \Delta$, where Δ is the Cartan simplex in $\mathfrak{t}_{\mathfrak{m}}$ (here we are using (5.2); note that $(L_{alg}G)^\tau \cap P^\tau = G^\sigma = K$). Hence the orbit space of the $(\Omega_{alg}G)^\tau$-action is a quotient of $K/C_K t_m \times \Delta$. As in the proof of (4.2), we see that the equivalence relation here coincides with that of Theorem 1.9. Hence the orbit space is G/K, as desired. To see that the action is free, we introduce the space of special paths $\mathscr{S}_{G/K}$ path of the form $f(e^{2\pi i t}) \exp tX$ with $f \in (\Omega_{alg}G)^\tau$ and $X \in \mathfrak{m}$. The proof now proceeds exactly as in (4.2); details are left to the reader. □

The other results of § 4 also go through: $\mathscr{S}_{G/K}$ is $(L_{alg}G)^\tau$—equivariantly homeomorphic to the building $\mathscr{B}_{G/K}$, and if $X, Y \in m$, $\exp X = \exp Y$ implies $\exp tX = f \exp tY$, where $f \in (\Omega_{alg}G)^\tau$.

§ 6. EXAMPLES

In this section we discuss six examples, the first four of which arise in the Bott periodicity theorems (§ 7). The first and last examples are discussed in some detail, the others are only sketched.

(6.1) $\Omega(SU(2n)/Sp(n))$. This is perhaps the simplest nonsplit example. $SU(2n)$ has an involution σ given by $\sigma(A) = J\bar{A}J^{-1}$, where J is the matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. The fixed group K is $Sp(n)$. The extension of σ to $SL(2n, \mathbf{C})$

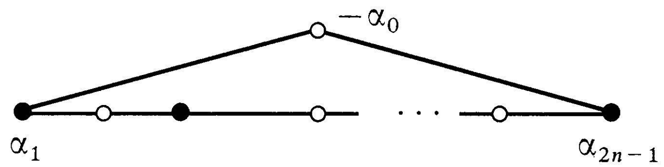
is given by the same formula, so the corresponding real form is $SL(n, \mathbf{H}) \equiv GL(n, \mathbf{H}) \cap SL(2n, \mathbf{C})$. For convenience we now make the obvious change of basis transforming J into a direct sum of 2×2 matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

In this basis $\mathfrak{t}_\mathfrak{m}$ consists of the diagonal matrices $a = \begin{pmatrix} a_1 & & & & \\ & a_1 & & & \\ & & \ddots & & \\ & & & a_n & \\ & & & & a_n \end{pmatrix}$ with the a_i pure imaginary.

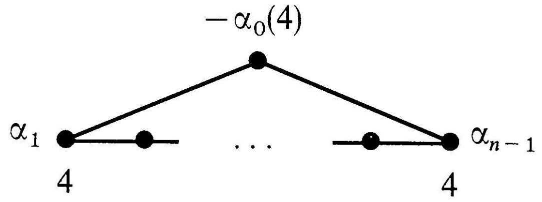
Hence $C_K\mathfrak{t}_\mathfrak{m} = \prod_1^n Sp(1)$, $N_K\mathfrak{t}_\mathfrak{m} = \sum_n \int Sp(1)$, and the relative Weyl group $W_{G,K}$ is $\sum_n$. The root systems are described as follows. In the usual notation, the root system Φ of $SU(2n)$ consists of

$$\{\pm (e_i - e_j): 1 \leqslant i, j \leqslant 2n, i \neq j\}.$$

Clearly $\Phi_0 = \{\pm (e_i - e_{i+1}): i \text{ odd}\}$. If $a \in \mathfrak{t}_\mathfrak{m}$ is as above, let $f_i(a) = a_i$. Then the restricted root system Σ consists of $\{\pm (f_i - f_j): 1 \leqslant i, j \leqslant n: i \neq j\}$, and so has type A_{n-1}. Moreover it is clear that the multiplicities are all equal to four. Thus the extended Satake diagram is



and the extended Dynkin diagram is



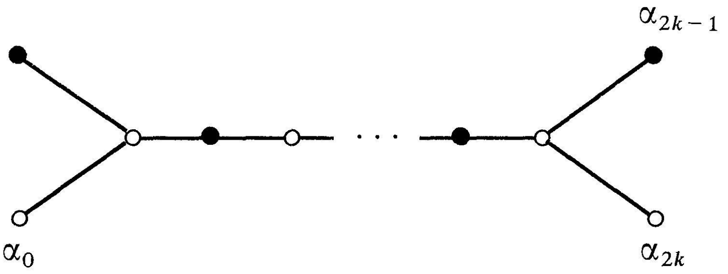
Note that the parabolic subgroup Q (obtained from the black nodes of the Satake diagram) is just the isotropy group of the standard flag $\mathbf{C}^2 \subset \mathbf{C}^4 \cdots \subset \mathbf{C}^{2n-2} \subset \mathbf{C}^{2n}$. The corresponding "quasi-Borel" subgroup Q^σ (minimal parabolic, in the standard terminology) is then the isotropy group of the complete quaternionic flag $\mathbf{H}^1 < \mathbf{H}^2 \cdots < \mathbf{H}^n$ (in $SL(n, \mathbf{H})$). The little K_α subgroups ($\alpha \in \Sigma$) are all $Sp(2)$'s.

Now consider the involution τ on $L_{alg}SL(2n, \mathbf{C}) = SL(2n, \mathbf{C}[z, z^{-1}])$. If $f(z) = \sum A_k z^k$, $(\tau f)(z) = \sum J\bar{A}_k J^{-1} z^k$. Hence the fixed group L^{τ}_{alg} is just $SL(n, \mathbf{H}[z, z^{-1}])$. Since we know that the affine Weyl group $\tilde{W}$ of type A_{n-1} has $P_{\tilde{W}/W}(t) = \prod_{i=1}^{n} (1-t^i)^{-1}$, the extended Dynkin diagram above shows immediately that $\Omega SU(2n)/Sp(n))$ has torsion—free homology, with Poincaré series $\prod_{i=1}^{n} (1-t^{4i})^{-1}$. For more applications of this approach, see [9] and § 7.

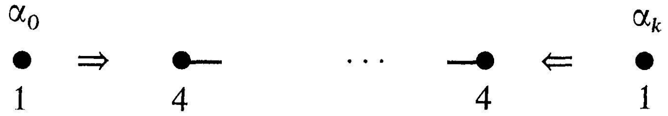
(6.2) $\Omega(SO(2n)/U(n))$. For convenience we take $n = 2k$, $k \geqslant 2$. Let J be as in (6.1) and define $\sigma(A) = JAJ^{-1}(A \in SO(2n))$. Then $K \cong U(n)$, embedded as the matrices $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. Now make the same change of basis as in (6.1). Then $\mathfrak{t}_{\mathfrak{m}}$ consists of matrices

$$A = \begin{pmatrix} A_1 & & & & & \\ & -A_1 & & & & \\ & & \ddots & & & \\ & & & A_k & & \\ & & & & & -A_k \end{pmatrix}$$

where $A_i = \begin{pmatrix} 0 & -a_i \\ a_i & 0 \end{pmatrix}$. Since the original root system Φ consists of $\{\pm e_i \pm e_j : 1 \leqslant i, j \leqslant n, i \neq j\}$, where e_i denotes projection on the ith 2×2 block in $\mathfrak{t}$, clearly $\sum$ has type C_k and consists of $\pm (f_i, -f_j)$, $\pm 2f_i$, where $f_i(A) = a_i$. We have $\Phi = \{\pm (e_i + e_{i+1}) : i \text{ odd}\}$ and $W_{G,K} = \sum_n \int \sum_2$. The simple roots $f_i - f_{i+1}$ have multiplicity 4, whereas $2f_i$ has multiplicity one. Thus the extended Satake diagram is



and the extended Dynkin diagram is



(Here the usual basis $e_1, -e_2, e_2 - e_3, \cdots e_{n-1} - e_n, e_{n-1} + e_n$ for Φ has been replaced by the basis

$$e_1 + e_2, -c_2 - e_3, e_3 + e_4, -e_4 - e_5, \cdots e_{n-1} + e_n, e_{n-1} - e_n .$$

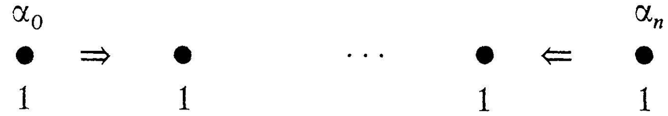
In particular the highest root is now $e_1 - e_2$).

(6.3) ($\Omega(SU(n)/SO(n))$. Here the involution on $SU(n)$ is $\sigma(A) = \bar{A}$. Hence we are in the split case and everything is transparent:

$$G_{\mathbf{R}} = SL(n, \mathbf{R}), (L_{alg}\, SL(n, \mathbf{C}))^{\tau} = SL(n, \mathbf{R}[z, z^{-1}]), \text{ etc.}$$

The Satake and Dynkin diagrams are just the Dynkin diagram for A_{n-1} (all Satake nodes white, all multiplicities equal one). For further details and applications, see [9].

(6.4) ($\Omega(Sp(n)/U(n))$. Embed $Sp(n)$ in $SU(2n)$ as usual and define $\sigma(A) = \bar{A}$. The fixed group is $U(n)$ embedded as matrices $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ with A, B real. Again we see that we are in the split case; the associated real form $G_{\mathbf{R}}$ is $Sp(n, \mathbf{R})$, L^{τ}_{alg} is $Sp(n, \mathbf{R}[z, z^{-1}]$, etc. The extended Dynkin diagram is



We can conclude e.g. that $\Omega Sp(n)/U(n)$ has mod 2 Poincaré series $\prod_{i=1}^{n} (1-t^{2i-1})^{-1}$ (cf. Theorem 5.9).

(6.5) ΩS^n. Assume $n = 2k + 1$; the case n even is similar. Define an involution σ on $SO(2k+1)$ by $\sigma(A) = \varepsilon A \varepsilon^{-1}$, where

$$\varepsilon = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

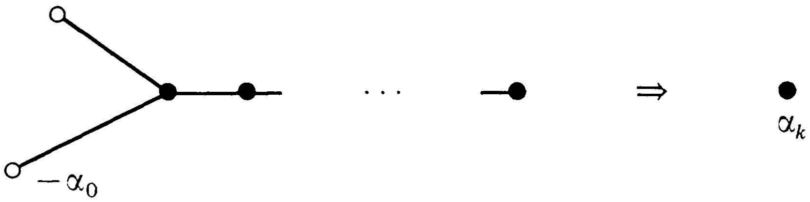
Then $K = S(O(1) \times O(2k)) \cong O(2K)$, so $K' = SO(2k)$. The corresponding real from $G'_{\mathbf{R}}$ consists of matrices $\begin{pmatrix} a_{11} & \cdots \\ \vdots & A \end{pmatrix}$ in $SO(2k+1, \mathbf{C})$ with a_{11} and A real and the remaining entries pure imaginary. In fact (as in easily checked) $G'_{\mathbf{R}} \cong SO(1, 2k)$. The torus $\mathfrak{t}_{\mathfrak{m}}$ is the set of matrices

$$\begin{pmatrix} 0 & -a & \\ a & 0 & \\ & & 0 \end{pmatrix}$$

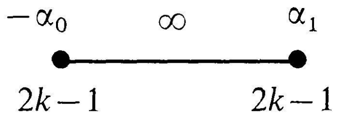
and hence the relative Weyl group has order 2

$$\left(\text{generated by} \begin{pmatrix} -1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}\right).$$

Using the usual notation for Φ, $\Phi_0 = \{\pm (e_i - e_j), \pm e_k : i, j, k \neq 1\}$. Thus Σ has type A_1 (no doubled roots) and the multiplicity of its one positive root is $2k - 1$. The extended Satake diagram is



and the extended Dynkin diagram is



(The symbol ∞ indicates that $s_0 s_1$ has infinite order.) The groups $\tilde{K}_0, \tilde{K}$, are both $SO(2k)$'s. In particular we obtain a model for ΩS^n with one cell in each dimension of the form $i(n-1)$.

(6.6) $\Omega \mathbf{C}P^{n-1}$. This example serves to illustrate two phenomena not encountered above: a nontrivial involution on the Satake diagram, and a restricted root system which is not reduced. Take $G = SU(n)$ and define $\sigma(A) = \varepsilon A \varepsilon$, where ε is as in (6.5). Thus $K = S(U(1) \times U(n-1))$ and $G/K = \mathbf{C}P^{n-1}$. The corresponding real form of $SL(n, \mathbf{C})$ is denoted $SU(1, n-1)$ and is described as in (6.5): matrices $\begin{pmatrix} a_{11} & \cdots \\ \vdots & A \end{pmatrix}$ in $SL(n, \mathbf{C})$ with a_{11}, A real and the remaining entries pure imaginary. The torus $\mathfrak{t}_{\mathfrak{m}}$ consists of matrices

$$\begin{pmatrix} 0 & a & \\ a & 0 & \\ & & 0 \end{pmatrix}$$

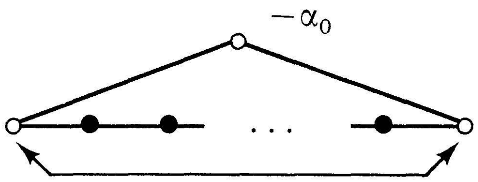
with a pure imaginary. Here we are taking as Cartan subalgebra in $\mathfrak{su}(n)$ the matrices

$$\begin{pmatrix} a & b & & & \\ b & a & & & \\ & & c_3 & & \\ & & & \ddots & \\ & & & & c_n \end{pmatrix}.$$

Using this Cartan subalgebra, a simple system of roots $\alpha_1, \cdots \alpha_{n-1}$ for Φ is given by the following table:

α_1	$2a + b - c_3$	
α_2	$-2a + c_3 - c_4$	
α_i	$c_{i+1} - c_{i+2}$	$(3 \leqslant i \leqslant n-2)$
α_{n-1}	$b + c_n$	

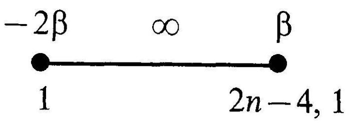
The highest root $\alpha_0 = \alpha_1 + \alpha_2 + ... + \alpha_{n-1}$ then takes the value $2b$. The action of σ on these roots is given by $\alpha_i \mapsto -\alpha_i (2 \leqslant i \leqslant n-2)$ and $\sigma\alpha_1 = \alpha_2 + \alpha_3 + \cdots + \alpha_{n-1}$. Thus Φ_0 is the span of $\alpha_2, \cdots \alpha_{n-2}$, and the extended Satake diagram is



Furthermore the restricted root system Σ has type BC_1 (type A_1 with doubled root). Indeed if β is defined by

$$\beta \begin{pmatrix} 0 & b & \\ b & 0 & \\ & & 0 \end{pmatrix} = b,$$

we see that β has multiplicity $2n - 4$ and 2β has multiplicity one (α restricts to 2β). Hence the extended Dynkin diagram is



Following the procedure discussed in § 5, we have at once that G_β is all of $SU(n)$, so $K_\beta = K \cong U(n-1)$. Note $K/C_K\mathfrak{t}_\mathfrak{m} = S^{2n-3}$. On the other hand $K_{2\beta} \cong SO(2)$ ($G_{2\beta}$ is the $SU(2)$ in the upper left corner). From the Dynkin diagram we conclude that our model for $\Omega \mathbf{C}P^{n-1}$ has one cell in each of the dimensions 0, 1, $2n-2$, $2n-1$, $4n-4$, $4n-3$, ... in other words, the cell series is $(1+t)(1+t^{2n-2})^{-1}$. (Recall that the affine Weyl group of type $\tilde{A}_1$ is just the free product $\mathbf{Z}/2 * \mathbf{Z}/2$, so that the Bruhat cells are indexed by $1, s_0, s_1s_0, s_0s_1s_0$, etc. By the above remarks, s_0 receives weight one and s_1 weight $2n-3$, hence our formula.)

§ 7. Bott Periodicity

Bott's theorem, in its original form [6], is a general statement about the range in which certain maps $K/L \xrightarrow{\varphi} \Omega G/K$ are homotopy equivalences. The periodicity theorems proper are then deduced from this, taking G, K, L to be suitable classical groups. In this section we derive a version of Bott's theorem by showing that in many cases the map φ is a homeomorphism onto a Schubert subspace of $\Omega(G/K)$; then one merely counts cells. In fact, in these cases we will be able to read off the desired range directly from the Dynkin diagram of G/K.

We assume that G is simple and simply-connected. (As usual, the essential point is that G/K is simply-connected; then we can if necessary replace G by its universal cover.) Let $\lambda: [0, 1] \to G$ be a path of the form $\lambda(t) = \exp tX$, where X belongs to the coweight lattice J_m. In otherwords, $X \in t_\mathfrak{m}$ and $\exp X$ is central in G. Then for all $k \in K$, the path $\varphi_\lambda \equiv \lambda k \lambda^{-1} k^{-1}$ actually lies in $(\Omega_{alg}G)^\tau$; see the proof of 4.2. Hence $\lambda \mapsto \varphi_\lambda$ defines a *Bott map* $K/C_k\lambda \xrightarrow{\varphi} (\Omega_{alg}G)^\tau$ $(\cong \Omega G/K)$. Identifying J_m with the group of paths λ as above, the most interesting λ are obviously the fundamental coweights ε_i dual to the simple restricted roots β_i: $\beta_j(\varepsilon_i) = \delta_{ij} (1 \leqslant i, j \leqslant l)$. Among these one may single out the very convenient class of *miniscule coweights*. These are the ε_i dual to a *miniscule root* β_i-i.e. a simple root which occurs with coefficient one in the highest root β_0. The miniscule coweights are precisely the nonzero elements of the coweight lattice which are also vertices of the Cartan simplex. They exist whenever the root system is reduced and not of type G_2, F_4 or E_8; in terms of the Dynkin diagram, they correspond to nodes on the ordinary diagram which are conjugate to the special node $-\alpha_0$ under an automorphism of the extended diagram. Thus for example in type A_n every simple root is miniscule, whereas the number of miniscule roots in types B_n, C_n, D_n, E_6, E_7 is respectively 1, 1, 3, 3, 1. Next, define the *distance* $d(s_i, s_j)$ between two elements of $\tilde{S}_{\mathbf{R}}$ (or nodes on the extended Dynkin diagram of G/K) as follows. Given a path p from s_i to s_j on the extended Dynkin diagram, let m_p be the sum of the multiplicities of the vertices of the path (including s_i and s_j). Then $d(s_i, s_j)$ is the minimal possible value of m_p (p ranging over all paths). For example, in the split case, with $m_\beta = 1$ for all simple restricted roots β, $d(s_i, s_j)$ is just the minimal number of vertices in a path linking s_i to s_j. (Arrows are ignored, and doubled or tripled edges in the diagram are counted as single edges.) We may now state our version of Bott's theorem:

(7.1) THEOREM. *Let ε_i be a miniscule coweight of the restricted root system Σ, and let $\varphi: K/C_K\varepsilon_i \to \Omega G/K$ be the Bott map associated to ε_i^{-1}. Then φ is an isomorphism on homotopy groups in dimensions less than $d(s_0, s_i) - 1$, and is an epimorphism in dimension $d(s_0, s_i) - 1$.*

(7.2) COROLLARY (Bott Periodicity). *There exist Bott maps of the following form, which are isomorphisms on homotopy through the indicated range of dimensions:*

(*a*)	$G^{\mathbf{C}}_{2n,2} \to \Omega SU(2n)$	$(2n)$
(*b*)	$SO(4n)/U(2n) \to \Omega_0 SO(4n)$	$(4n-4)$
(*c*)	$U(2n)/Sp(n) \to \Omega SO(4n)/U(2n)$	$(4n-4)$
(*d*)	$G^{\mathbf{H}}_{2n,n} \to \Omega SU(4n)/Sp(2n)$	$(4n+2)$
(*e*)	$Sp(n)/U(n) \to \Omega Sp(n)$	$(2n)$
(*f*)	$U(n)/O(n) \to \Omega Sp(n)/SU(n)$	(n)
(*g*)	$G^{\mathbf{R}}_{2n,n} \to \Omega SU(2n)/SO(2n)$	$(n-1)$

Proof of Corollary. We need only exhibit miniscule coweights ε_i such that $d(s_0, s_i) - 2$ is the number indicated and $K/C_K\varepsilon_i$ is as shown. We will do this for (*c*) and (*d*) and leave the rest of the fun to the reader (see § 6). In case (*d*), we have seen that Σ has type A_{4n-1} and hence every simple root is miniscule; we also know the multiplicities all equal four. Taking $\varepsilon_i = \varepsilon_n$, we obviously have $d(s_0, s_n) = 4n + 4$. In case (*c*), Σ has type C_{2n}; there is one miniscule root α_{2n}. From (6.2) we compute $d(s_0, s_{2n}) = 4n - 2$. □

Proof of (7.1). The proof is an easy generalization of that of Propositions 2.2 and 2.6 in [25] (note, however, that $d(s_i, s_j)$ is defined somewhat differently there). Therefore it will only be sketched. First of all, consider the set of restricted roots β such that $\beta(\varepsilon) = 0$. This set is spanned by the set I of simple roots it contains, and if I' is the corresponding set in S (as usual), $C_G\varepsilon =$. Thus $C_K\varepsilon = (C_G\varepsilon)^\sigma = K_I$. Since K_I is a maximal compact subgroup of the parabolic $\mathcal{O}_I((=(P_{I'})^\sigma)$, the Iwasawa decomposition $\mathcal{O}_I = K_I Q$ shows that $K/C_K\varepsilon = G_{\mathbf{R}}/\mathcal{O}_I$. Since $(\Omega_{alg})^\tau = \tilde{G}_{\mathbf{R}}/P^\tau$, the Bott map can be thought of as a map $G_{\mathbf{R}}/\mathcal{O}_I \to \tilde{G}_{\mathbf{R}}/P^\tau$. To describe this map in terms of Bruhat cells we need to alter it slightly. First, let $y_i = \varepsilon_i w$, where $w = w_{[i]} w_0 \in W_{\mathbf{R}}$. Here $W_{[i]}$ denotes the maximal length element of $W_{[i]}$, where $[i] = S_{\mathbf{R}} - \{i\}$. (This definition is due to Iwahori and Matsumoto [16], among other things it provides a splitting of the projection $\tilde{W} \to \tilde{W}/\tilde{W}$.) Then the map $\varphi' : K/C_K\varepsilon \to (L_{alg}G)^\tau/K = (\Omega_{alg}G)^\tau$ given by $k \mapsto \mu_i^{-1} k \mu_i$ is homotopic to φ, since $\varphi' = w^{-1}\varphi$ and K is connected. Hence in the proof we may replace φ by φ'. The point of this is:

(7.3) LEMMA.

(a) *The map* $\Theta : f \mapsto \mu_i^{-1} f \mu_i$ *defines an automorphism of* $\tilde{G}_{\mathbf{C}}$ *preserving* $\tilde{G}_{\mathbf{R}}$.

(b) $\Theta : \tilde{G}_{\mathbf{R}} \to \tilde{G}_{\mathbf{R}}$ *preserves* $\tilde{Q}$, *and in fact permutes the simple roots (defining an automorphism of the extended Dynkin diagram). In particular* $\mu_i \cdot (1, \beta_0) = (0, -\beta_i)$.

(c) $\Theta \mid_{G_{\mathbf{R}}}$ *induces an embedding* $G_{\mathbf{R}}/\mathcal{O}_I \to \tilde{G}_{\mathbf{R}}/P^\tau$, *which corresponds to* φ' *and is a homeomorphism onto a Schubert subspace.*

Remarks. In (a) we have identified $\tilde{G}_{\mathbf{C}}$ with the group of paths: $[0, 1] \to G_{\mathbf{C}}$ of the form $f(e^{2\pi i t})$, where $f : S^1 \to G_{\mathbf{C}}$ is algebraic. In (b), the automorphism of the Dynkin diagram preserves multiplicities.

It remains to show that every cell not in the image of φ' has dimension at least $d(s_0, s_j)$. Now Θ preserves the simple reflections $\tilde{S}_{G/K}$, with $\Theta(s_i) = s_0$, and clearly the cells which are in the image of φ' are precisely the E_w such that $w \in W^{S_{\mathbf{R}}}$ and $\Theta(s_0)$ does not occur in a reduced expression for w. Since every such expression must begin on the right with s_0, a moments reflection should convince the reader that the minimal dimension of a cell involving $\Theta(s_0)$ is $d(s_0, \Theta(s_0))$. Since

$$d(s_0, \Theta(s_0)) = d(\Theta^{-1}(s_0), s_0) = d(s_0, s_i),$$

this completes the proof. □

§ 8. Appendix: Real Forms and the Generalized Bruhat Decomposition

Let $G_{\mathbf{C}}$ be a reductive complex algebraic group, as usual, and let $P = P_I$, $Q = P_J$ be parabolic subgroups. Let $H_{\mathbf{C}}$ be "the" Levi factor of P with maximal compact subgroup H. Explicitly, $H_{\mathbf{C}}$ is the (closed, connected) subgroup whose Lie algebra is generated by $\mathfrak{t}_{\mathbf{C}}$ and the $X_{-\alpha_{\pm\alpha}}$, $\alpha \in I$. We have $P = H_{\mathbf{C}} U_I$, where the unipotent radical U_I corresponds to the positive roots not in the span of I.

(8.1) Theorem. *The P-orbits in* $G_{\mathbf{C}}/Q$ *are holomorphic vector bundles over flag varieties of* $H_{\mathbf{C}}$.

Theorem (8.1) is certainly well known, although not so easy to find in the literature. In this section we will prove (8.1) and its loop group analogue in a more explicit form, and show how one may easily deduce the Bruhat decomposition for real forms from this. (The proofs of various technical lemmas about root systems will be omitted. The details are somewhat tedious, but not difficult.)

(8.2) Lemma. *Each* $W_I - W_J$ *double coset in* W *contains a unique element* w *of minimal length. For such* $a\ w$ *we have*

(a) $\{x \in W_I : w^{-1}xw \in W_J\} = W_K$, *where* $K = \{s \in I : w^{-1}sw \in J\}$.

(b) *each* $x \in W_I\, w\, W_J$ *has a unique factorization of the form* $x = vwy$, *with* $v \in (W_I)^K$, $y \in W_J$. *Furthermore* $l(vwy) = l(v) + l(w) + l(y)$, *(in particular* $vw \in W^J$*).* □

Let w be minimal as in (8.2), and let $E = \{h \in H_{\mathbf{C}} : w^{-1}\, h\, w \in Q\}$ (i.e., E is the isotropy group of wQ in $H_{\mathbf{C}}$).

(8.3) Lemma. E *is a parabolic subgroup of* $H_{\mathbf{C}}$, *and its Levi factor* $F_{\mathbf{C}}$ *normalizes* U_w. □

We recall here that $U_w = \{u \in U : w^{-1}uw \in U^-\}$. In the present situation it is easy to see that $U_w \leqslant U_I$, and $w^{-1}U_w w \leqslant U_I^-$. The proof of (8.3) then reduces to a simple calculation in the root system. Now form the balanced product $H_{\mathbf{C}} \times_E U_w$, where E acts on U_w via the projection $E \to F_{\mathbf{C}}$. Since $\exp : \mathfrak{u}_w \to U_w$ is an ad-equivariant isomorphism of varieties,

$H_{\mathbf{C}} \times_E U_w$ is an algebraic vector bundle over the flag variety $H_{\mathbf{C}}/E$. Of course we also have $H_{\mathbf{C}}/E = H/F$ and $H_{\mathbf{C}} \times_E U_w = H \times_F U_w$ (by the Iwasawa decomposition).

(8.4) THEOREM. *The map* $\varphi: H_{\mathbf{C}} \times_E U_w \to PwQ/Q$ *given by* $(h, u) \mapsto huwQ/Q$ *is an isomorphism of varieties.*

Proof. Clearly φ is well—defined and surjective. To see that φ is injective, note that the Bruhat decomposition of $H_{\mathbf{C}}/E$ lifts to a cell decomposition of $H_{\mathbf{C}} \times_E U_w$; the cells are of the form $(U_v v) \times (U_w w)$, where v ranges over $(W_I)^K$. By Lemma (8.2), the vw are distinct elements of W^J, so φ maps cells to Bruhat cells. Finally, (8.2) and the Steinberg lemma show that φ is injective on each cell. □

Example. Let $G_{\mathbf{C}} = GL(n, \mathbf{C})$, so $W = \sum_n$ and $S = (s_1, \cdots s_{n-1})$ as usual. Take $I = J = S - \{s_k\}$, so $G_{\mathbf{C}}/Q$ is the Grassman manifold of k-planes in n-space, and $W_I = \sum_k \times \sum_{n-k}$. The $(I-I)$-minimal elements are precisely the shuffles σ_i defined by $\sigma_i(r) = r$ (if $1 \leqslant r \leqslant i$), $\sigma_i(r) = k + r - i (i+1 \leqslant r \leqslant k)$; here $i \leqslant k$ and $k - i \leqslant n - k$. Note σ_i has length $(k-i)^2$. The $P(=Q)$-orbit of σ_i is $\{W \in G_{n,k}: \dim W \cap \mathbf{C}^k = i\}$, where $\mathbf{C}^k$ is the span of the first k basis vectors. This orbit can then be identified with the vector bundle $\hom(\gamma_{n-k,k-i}, \gamma^{\perp}_{k,i})$ over $G_{k,i} \times G_{n-k,k-i}$ (γ denoting the canonical bundle).

Now suppose given an involution σ on $G_{\mathbf{C}}$ (in normal form) with $G_{\mathbf{R}} = (G_{\mathbf{C}})^{\sigma}$, etc. We take I corresponding to the black nodes on the Satake diagram—i.e., I corresponds to the simple roots α such that $\sigma\alpha = -\alpha$. Also take $I = J$, so $Q = P_I$. We have $B_{\mathbf{R}} = Q^{\sigma}$ (by definition) and $W_{\mathbf{R}} = W^{\sigma}/W_I$ (W_I is usually denoted W_0). Note that σ preserves Q and hence permutes the $Q - Q$ double cosets.

(8.5) THEOREM.

(a) *If* $w \in W^{\sigma} \cap W^I$, *then* $QwQ = BwQ = U_w wQ$ *and* σ *acts on* U_w *as a conjugate linear involution.*

(b) *If* $w \notin W^{\sigma}$, $QwQ \cap G_{\mathbf{R}}$ *is empty.*

(8.6) COROLLARY (Bruhat decomposition). $G_{\mathbf{R}} = \coprod_{w \in W_{\mathbf{R}}} B_{\mathbf{R}} w B_{\mathbf{R}}$.

Proof. Note that on $H_{\mathbf{C}}/C(H_{\mathbf{C}})$, σ is the compact involution. In particular σ is the identity on $H/C(H)$; in fact $H = (C_K t_m)(C(H))$. It follows that $hwQ = wQ$ for $h \in H$, and hence $QwQ = UwQ = U_w wQ$. A calculation with roots shows that $\sigma(U_w) = U_w$, and (a) follows. Now consider a fixed w

of minimal length in $W_I w W_I$. To prove (b), we may as well assume that σ preserves QwQ; i.e., $\sigma(w) = awb$ with $a, b \in W_I$. We identify the orbit QwQ with $H \times_F U_w$ as in Theorem (8.4).

(8.7) LEMMA.

(a) $a \in N_H F$, *and* a *is well defined* mod F.

(b) $a^{-1}\sigma(U_w)\, a = U_w$. □

It now follows that σ acts as a conjugate linear bundle automorphism:

$$\sigma(huwQ) = hz\sigma(u)awQ = hzau'Q = hazu'Q,$$

where $u' = a^{-1}\sigma(u)a$ and $z \in C(H)$. Furthermore the action on the base H/F is given by $\sigma(hF) = haF$. Hence either σ acts freely on the base, and hence freely on the orbit, or else $a \in F$. In the latter case $\sigma(wW_I) = wW_I$. But it is a (trivial) exercise in linear algebra to show that this implies $\sigma(w) = w$. □

Example. Consider the involution $\sigma(A) = J\bar{A}J^{-1}$ on $SU(4)$, as in § 6. Q is the stabilizer of the standard 2-plane in $\mathbf{C}^4$ and $H = S(U(2) \times U(2))$. The minimal length elements (with respect to $I - I$, where $I = (s_1, s_3)$) are $1, s_2, s_2 s_1 s_3 s_2$ (shuffles as in the preceding example). The corresponding Q-orbits are respectively a point, a line bundle over $\mathbf{C}P^1 \times \mathbf{C}P^1$, and a cell of complex dimension four. The action of σ on $\mathbf{C}P^1 \times \mathbf{C}P^1$ is the obvious one on each factor, arising from the quaternionic "j" acting on complex lines in $\mathbf{H}$, and obviously is free. Taking fixed points yields the usual cell decomposition of the 4-sphere $\mathbf{H}P^1$.

Now consider our algebraic loop group $\tilde{G}_{\mathbf{C}}$. For simplicity we consider only parabolics P_I, P_J with $I, J \leqslant S$ (i.e. $P_I, P_J \leqslant P = G_{\mathbf{C}[z]}$), although this is not really necessary.

(8.8) THEOREM. *The P_I-orbits in $\tilde{G}_{\mathbf{C}}/P_J$ are holomorphic vector bundles over flag varieties of $H_{\mathbf{C}}$.* □

Here we note that although our notation is slightly ambiguous—in (8.8) P_I is a parabolic subgroup of $\tilde{G}_{\mathbf{C}}$, but it could also denote a parabolic in $G_{\mathbf{C}}$—the Levi factor $H_{\mathbf{C}}$ is the same for either interpretation. In any case the proof is identical to the proof of the classical case, with the affine root system replacing the ordinary root system. In particular the analogues of (8.2), (8.3), and (8.4) hold.

Example. Consider the P-orbits of $\tilde{G}_{\mathbf{C}}/P = \Omega_{alg}G$ (G simply-connected). These are indexed by homomorphisms $\lambda: S^1 \to T$ that lie in the closure of the dominant Weyl chamber ($\alpha(\lambda) \geqslant 0$ for all $\alpha \in \Phi^+$), and are precisely the stable manifolds of the energy flow on ΩG ([28]). The Levi factor $H_{\mathbf{C}}$ is just $G_{\mathbf{C}}$ in this case, so $H = G$, and $P\lambda P/P$ is a vector bundle over $G/C_G\lambda$. Now $W\lambda W = \coprod_{\lambda' \sim \lambda} \lambda' W$, where $\sim$ means W-conjugate. Hence, although λ will not be minimal in $W\lambda W$, the formula of Iwahori and Matsunoto shows that the minimal element has the form λw, and has length $\sum_{\alpha \in \Phi^+} \alpha(\lambda) - |\{\alpha > 0 : \alpha(\lambda) \neq 0\}|$. Hence this length is the complex dimension of the vector bundle in question, and one can even determine the bundle explicitly. For example, suppose $G = SU(n)$. Then λ corresponds to a sequence of integers $(b_1, \ldots, b_n)$ with $\sum b_i = 0$ and $b_1 \geqslant b_2 \ldots \geqslant b_n$. Write this sequence in the form $(a_1, a_1, \ldots a_r, a_r)$, where there are i_1 entries a_1, i_2 entries a_2, etc. Then $G/C_G\lambda$ is the flag variety $U(n)/U(i_1) \times \ldots \times U(i_r)$. Over this there are r canonical bundles ξ_k of dimension i_k; let $\xi_{kl} = \hom(\xi_l, \xi_k)$. Now $\sum_{\alpha > 0} \alpha(\lambda) = \sum_{k<l} (a_k - a_l) i_k i_l$, and $|(\alpha > 0 : \alpha(\lambda) \neq 0)| = \sum_{k<l} i_k i_l$. This suggests that the bundle is $\oplus_{k<l} (a_k - a_l - 1) \xi_{kl}$, and indeed this is easily verified. For the ξ_{kl} are precisely the irreducible components of the adjoint action of $C_G\lambda$ on the Lie algebra of the unipotent radical of the corresponding parabolic (i.e. the Lie algebra spanned by the X_α with $\alpha > 0, \alpha(\lambda) \neq 0$). Then one can check that $U_{\lambda w}$ corresponds to the strictly positive roots (n, α) (i.e. $n \geqslant 1$) such that $\lambda^{-1} \cdot (n, \alpha) = (n - \alpha(\lambda), \alpha)$ is strictly negative (i.e. $n - \alpha(\lambda) \leqslant -1$). Furthermore since $C_G\lambda$ consists of constant loops, it preserves the sum of the root subalgebras of fixed height n. Hence each ξ_{kl} (thought of as a representation of $C_G\lambda$) occurs in $U_{\lambda w}$ with multiplicity $a_k - a_l - 1$, which proves our assertion.

Finally, consider the involution t on $\tilde{G}_{\mathbf{C}}$. Theorem (8.5) and its proof carry over without difficulty, and we obtain:

(8.9) THEOREM. $\tilde{G}_{\mathbf{R}} = \coprod_{w \in W_{\mathbf{R}}} \tilde{B}_{\mathbf{R}} w \tilde{B}_{\mathbf{R}}$.

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Stephen A. Mitchell

Department of Mathematics
University of Washington
Seattle, WA 98195