

# Remarks

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## REMARKS

There are several proofs and several formulations of this result ( $S(K)$  when  $K$  is dyadic) in the literature. We shall briefly indicate why these formulations are, with one exception, equivalent to the above one.

1. *T. Yamada*, [Y], p. 88. One formulation of Yamada's theorem is that  $S(K)$  is non-trivial iff there is a root of unity  $\zeta$  such that the inertia group of the extension  $\mathbf{Q}_2(\zeta)/K$  is non-cyclic. The inertia group of  $\mathbf{Q}_2(\zeta)/K$  is the image of the inertia group of  $\mathcal{G}(\mathbf{Q}_2^c/K)$ , namely  $\mathcal{G}(\mathbf{Q}_2^c/K_{nr})$ . The latter group is of the form  $\hat{\mathbf{Z}}_2 \times (\mathbf{Z}/2)$  or  $\hat{\mathbf{Z}}_2$ , depending on whether or not  $\sigma_{-1} \in \mathcal{G}(\mathbf{Q}_2^c/K)$ . If  $\sigma_{-1} \notin \mathcal{G}(\mathbf{Q}_2^c/K)$ , then it follows that the inertia group of  $\mathbf{Q}_2(\zeta)/K$  is always cyclic. Suppose  $\sigma_{-1} \in \mathcal{G}(\mathbf{Q}_2^c/K)$ . Then  $\mathcal{G}(\mathbf{Q}_2^c/K_{nr}) = \hat{\mathbf{Z}}_2 \times \mathbf{Z}/2$  where the first factor is topologically generated by  $\sigma_5^{2^k}$  for some  $k \geq 0$  and  $\mathbf{Z}/2$  is generated by  $\sigma_{-1}$ . If we choose  $\zeta$  to have order divisible by a power of 2 large enough so that  $\sigma_5^{2^k}(\zeta) \neq \zeta$ , then it is clear that the inertia subgroup of  $\mathbf{Q}_2(\zeta)/K$  is not cyclic. Thus the inertia group of  $\mathbf{Q}_2(\zeta)/K$  is non-cyclic iff  $\sigma_{-1} \in \mathcal{G}(\mathbf{Q}_2^c/K)$ , and so Yamada's criterion is equivalent to mine.

2. *U. Fontaine*, [F], Cor. 2', p. 138. The result is:  $S(K)$  is non-trivial iff  $\varepsilon_4 \notin K$ . This is easily seen to be inequivalent to the other formulations. As an example, let  $K$  be the subfield of  $\mathbf{Q}_2(\varepsilon_{16})$  fixed by the automorphism  $\sigma_{-1}\sigma_5^2$ . Then  $\varepsilon_4 \notin K$  and  $\sigma_{-1} \notin \mathcal{G}(\mathbf{Q}_2^c/K)$ .

3. *G. J. Janusz*, [J], p. 543. Let  $h$  be the smallest integer  $\geq 2$  such that there is an odd integer  $c \geq 1$  with the property that  $\mathbf{Q}_2(\varepsilon_{2^h}, \varepsilon_c)$  contains  $K$ . Then Janusz' theorem is the following:

$S(K)$  is non-trivial iff there is an odd integer  $n$  with the following properties:

- (i)  $K(\varepsilon_4)/K$  is ramified.
- (ii)  $K(\varepsilon_{4n}) = \mathbf{Q}_2(\varepsilon_{2^h}, \varepsilon_n)$ .
- (iii)  $(K(\varepsilon_n): K) = 2^r w$ , where  $w$  is odd and  $r \geq 1$ .
- (iv) The automorphism of order 2 in  $\mathcal{G}(K(\varepsilon_{4n})/K(\varepsilon_n))$  carries  $\varepsilon_{2^h}$  to  $\varepsilon_{2^h}^{-1}$ .
- (v) If  $r \leq h - 1$ , then any root of unity in  $K(\varepsilon_{4n})$  whose order divides  $2^{h-r+1}$  already lies in  $K(\varepsilon_4)$ .

It can be shown that the conditions (iii) and (v) can be omitted. Indeed suppose that we are given an odd integer  $n$  such that (i), (ii), and (iv) are

satisfied. Let the the residue class field of  $K(\varepsilon_n)$  have  $2^k$  elements. Set  $n' = (2^k)^{2^h} - 1$ . Then  $n \mid n'$ ,  $n'$  is odd, and  $K(\varepsilon_{n'})/K(\varepsilon_n)$  is unramified of degree  $2^h$ . Consider the conditions (i)-(v) with  $n'$  instead of  $n$ . Then (i) is unchanged, (ii) holds because  $n \mid n'$ , (iii) holds trivially and (v) holds vacuously because  $2^h \mid (K(\varepsilon_{n'}) : K)$ . Finally  $K(\varepsilon_{n'}) \cap K(\varepsilon_4) = K$  since one is ramified and the other is not, so the non-trivial automorphism of  $K(\varepsilon_{4n})/K(\varepsilon_n)$  is the restriction of that of  $K(\varepsilon_{4n'})/K(\varepsilon_{n'})$ , so (iv) holds also for  $n'$ .

We can deduce from this abbreviated form of Janusz' theorem that it is equivalent to Yamada's. Suppose that Janusz' conditions are satisfied, and consider the extension  $\mathbf{Q}_2(\varepsilon_{2^{h+1}}, \varepsilon_n)/K$ . The inertia subgroup of its Galois group is  $\mathcal{g} = \mathcal{G}(\mathbf{Q}_2(\varepsilon_{2^{h+1}}, \varepsilon_n)/K(\varepsilon_n))$ , a group of order 4. Suppose that  $\rho$  is an extension of the non-trivial automorphism of  $\mathbf{Q}_2(\varepsilon_{2^h}, \varepsilon_n)/K(\varepsilon_n)$  to  $\mathbf{Q}_2(\varepsilon_{2^{h+1}}, \varepsilon_n)$ , so  $\rho \in \mathcal{g}$ . By condition (iv), there is an integer  $a \equiv -1 \pmod{2^h}$  such that  $\rho(\varepsilon_{2^{h+1}}) = \varepsilon_{2^{h+1}}^a$ . It follows that  $\rho^2$  is the identity. Thus  $\mathcal{g}$  is non-cyclic. Conversely suppose that there is an extension  $\mathbf{Q}_2(\zeta)/K$  whose inertia subgroup  $\mathcal{g}$  is non-cyclic. As we saw in 1., this means that  $\sigma_{-1}$  is in the Galois group of  $\mathbf{Q}_2^c/K$  and so its restriction (which we also call  $\sigma_{-1}$ ) is in  $\mathcal{G}(\mathbf{Q}_2(\varepsilon_{2^h}, \varepsilon_c)/K)$  and is non-trivial. Its fixed field contains  $K(\varepsilon_c)$ ; by Lemma 3.3 of [J],  $K(\varepsilon_c, \varepsilon_4) = \mathbf{Q}_2(\varepsilon_{2^h}, \varepsilon_c)$  and so the fixed field is *exactly*  $K(\varepsilon_c)$ . Thus both (iv) and (ii) are also fulfilled. (i) holds by Lemma 1.

4. *F. Lorenz*, [L], p. 463. His condition for *non-triviality of  $S(K)$*  is that  $-1$  is a norm in the extension  $K/\mathbf{Q}_2$ . The norm residue symbol in the extension  $\mathbf{Q}_2^c/\mathbf{Q}_2$  sends  $-1$  to  $\sigma_{-1} \in \mathcal{G}(\mathbf{Q}_2^c/\mathbf{Q}_2)$ . Thus it follows from [S], pp. 204-205, that  $-1$  is a norm in  $K/\mathbf{Q}_2$  iff  $\sigma_{-1} \in \mathcal{G}(\mathbf{Q}_2^c/K)$ .

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