

4. Properness

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

LEMMA 2.1. Let A, B be metric spaces, with $A \neq \emptyset$ and B connected. Let $P: A \rightarrow B$ be a continuous map. Assume:

- (i) P is open,
- (ii) P is proper, that is, for any compact subset K in B , $P^{-1}(K)$ is compact. Then P is surjective.

Proof. We only need to prove that $P(A)$ is closed. Let b_0 be a point in $\overline{P(A)}$. Since B is a metric space, there exists a sequence $(b_i)_{i>0}$ in $P(A)$ converging to b_0 . The subset $K = \{b_0, b_1, b_2, \dots\}$ is compact, hence so is $PP^{-1}(K)$. The latter contains $b_1, \dots, b_i, \dots$, hence b_0 , and it is obviously contained in $P(A)$. Q.E.D.

In order to make use of this lemma, we shall need some inverse function theorem for (i), and some *a priori* estimates for (ii).

3. LOCAL INVERSION

THEOREM 3.1. Let X be a smooth compact manifold, V and W smooth vector bundles on X , U an open set in $C^\infty(X, V)$, and $P: U \rightarrow C^\infty(X, W)$, a smooth nonlinear elliptic partial differential operator. Let A and B be LCFC submanifolds of U and of $C^\infty(X, W)$ respectively, such that the restriction P_A of P to A , sends A into B . Then the Jacobian criterion holds for P_A , namely, if the derivative of $P_A: A \rightarrow B$ is invertible at $\varphi_0 \in A$, then P_A is a local diffeomorphism near φ_0 .

This is a convenient variant of the Nash-Moser theorem (e.g. [14]) regarding suitable restrictions of elliptic operators. It is established in a separate paper [11] (see also [22]). It relies only on the classical (Banach) inverse function theorem combined with *elliptic regularity*.

Remark 3.2. The Nash-Moser theorem has been studied by many authors, see the bibliography below and further references in [14] [15] [25].

4. PROPERNESS

In view of (2), theorem 3.1 implies that P_λ is open. We want to apply lemma 2.1 in order to prove that P_λ is surjective from A_λ to B_λ . Since $P_\lambda(A_\lambda) \neq \emptyset$ (it contains 0), and since B_λ is connected, this amounts to proving that P_λ is *proper*. Let us explain why *a priori* estimates imply properness.

Concerning subsets in A_λ we have

PROPOSITION 4.1. *A subset S in A_λ is relatively compact in A_λ iff its closure $\bar{S}$ in $C^\infty(X)$ lies inside A_λ and S is bounded in $C^\infty(X)$.*

This readily follows from Ascoli theorem which implies the well-known fact [12] (p. 231) that in $C^\infty(X)$ (and in any closed LCFC submanifold of $C^\infty(X)$, such as B_λ , as well) bounded subsets are relatively compact and vice-versa; hence, compact subset of A_λ are nothing but bounded closed strictly interior subsets of A_λ . Explicitely, let us state the

COROLLARY 4.2. *A closed subset S in A_λ is compact if and only if there exists a sequence $(C_i), i \in \mathbb{N}$, of positive numbers, such that for any φ in S the following estimates hold:*

$$\begin{aligned} \|(g')^{-1}\| &=: \sup_X |(g')^{-1}| \leq C_0, \\ \forall i \in \mathbb{N}, \quad \|D^i \varphi\| &=: \sup_X |D^i \varphi| \leq C_i, \end{aligned}$$

where $|\cdot|$ denotes some natural norms of tensors in the original metric g , and $D =: (\nabla, \bar{\nabla})$ is the total covariant differentiation with respect to the metric g .

Proof. Indeed S is closed and bounded. Moreover, since for $\varphi \in S$,

$$\|(g')^{-1}\| \leq C_0$$

all the eigenvalues of $(g')^{-1}$ (which are positive) are uniformly bounded from above, hence those of g' are uniformly bounded from below, in other words:

$$\exists \varepsilon > 0, \quad \forall \varphi \in S, \quad g' \geq \varepsilon g,$$

or equivalently $\bar{S}$ lies strictly inside A_λ . Q.E.D.

In the next sections we will show that if f belongs to some compact (i.e. bounded and closed) subset K of B_λ , defined by a sequence $(K_i), i \in \mathbb{N}$, such that $\|D^i f\| \leq K_i$, then for $\varphi \in A_\lambda$ satisfying $P_\lambda(\varphi) = f$, the following *a priori* estimates hold:

$$\|\varphi\| \leq C_0, \quad \forall i \in \mathbb{N}, \quad \|D^i \nabla \bar{\nabla} \varphi\| \leq C_{i+2}.$$

These estimates imply that P_λ is proper, i.e. that $S = P_\lambda^{-1}(K)$ is compact, according to the following

PROPOSITION 4.3. *Let S be a closed subset in A_λ . Suppose that there exists a sequence $(C_i), i \in \mathbb{N}$, such that for any φ in S , the following estimates hold:*

$$\| \varphi \| \leq C_0, \quad \| P_\lambda(\varphi) \| \leq C_0, \quad \forall i \in \mathbf{N}, \quad \| D^i \nabla \bar{\nabla} \varphi \| \leq C_{i+2}.$$

Then S is compact.

Proof. The first two estimates imply a uniform estimate

$$| \operatorname{Log} \det (g' g^{-1}) | \leq E.$$

The estimate on $\| \nabla \bar{\nabla} \varphi \|$ yields another one:

$$\| g' \| \leq F.$$

These two estimates yield

$$\| (g')^{-1} \| \leq G.$$

Now from $\| D^i \nabla \bar{\nabla} \varphi \| \leq C_{i+2}$ we infer

$$\| D^i \Delta \varphi \| \leq \tilde{C}_{i+2}$$

since D and g^{-1} commute (Δ denotes the Laplacian in the metric g). As Δ performs a continuous linear automorphism of the Fréchet space of smooth functions *with zero average* (by Fredholm theory), the Closed Graph Theorem implies the missing estimates. Q.E.D.

Remark 4.4. Actually we have been considering two *gradings* of $C^\infty(X)$ [14]. The usual one, namely the one defined, $\forall u \in C^\infty(X)$, by

$$\| u \|_0 = \sup_X | u |,$$

$$\| u \|_i = \| u_i \|_{i-1} + \| D^i u \|, \quad i \geq 1,$$

and another one, well-adapted here since the true unknown is a Kähler metric, defined by

$$\| u \|_0^* = \| u \|_0, \quad \| u \|_1^* = \| u \|_1,$$

$$\| u \|_i^* = \| u \|_{i-1}^* + \| D^{i-2}(\nabla \bar{\nabla} u) \|, \quad i \geq 2.$$

Although it is unnecessary for the purpose of proposition 4.3, it can be shown globally (without Schauder theory) that these two gradings are *tamely* equivalent [14] of degree 2 and base 0 [10] (section 5). Hence, they define the same topology.