

## §3. Loop Groups

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$x$  is a *cut point* (with respect to  $p$ ) if there is a geodesic from  $p$  to  $x$  that minimizes arc length up to  $x$  but no further. The *cut locus* is the set of cut points. Similarly a vector  $X$  in the tangent space  $T_p$  is a *tangent cut point* if  $\exp_p X$  is a cut point along the geodesic  $\exp_p(tX)$ . The *tangent cut locus* is the set of all such points in  $T_p$ , and is homeomorphic to the unit sphere in  $T_p$ . When  $M = G/K$  we take  $p = 1$ .

(2.26) THEOREM. *Let  $G/K$  be a simply-connected symmetric space, with  $G$  simple. Then the tangent cut locus is precisely the  $K$ -orbit in  $\mathfrak{m}$  of the outer wall of the Cartan simplex  $\Delta_{\mathfrak{m}}$ . It is therefore canonically identified with the topological building of the associated real form  $G_{\mathbf{R}}$ .*

As usual, the assumption  $G$  simple is just for convenience. We sketch the proof: the first assertion is a fairly easy consequence of Theorem (1.8), and is left to the reader. Now consider the building  $\mathcal{B}_{G_{\mathbf{R}}}$ . It is a quotient space of  $G_{\mathbf{R}}/B_{\mathbf{R}} \times \Delta_0 = K/C_K t_{\mathfrak{m}} \times \Delta_0$ , where  $\Delta_0$  is a simplex of dimension  $(\text{rank } G/K) - 1$ ; we take  $\Delta_0$  to be the outer wall of  $\Delta_{\mathfrak{m}}$ . For each  $I \leq S_{G/K}$ , let  $\Delta_I$  temporarily denote the corresponding face of  $\Delta_0$ ; i.e.  $\{X \in \Delta_0 : \alpha_i(X) = 0 \ \forall i \in I\}$ . Then the  $K$ -orbit of  $\Delta_0$  in  $\mathfrak{m}$ ,  $K\Delta_0$ , is also a quotient of  $K/C_K t_{\mathfrak{m}} \times \Delta_0$ . The relations are  $(k_1 X) \sim (k_2 X)$  if  $X \in \Delta_I$  and  $k_1 = k_2 \text{ mod } K_I$ . But  $K_I = K \cap \mathcal{O}_I$ , so these relations are identical to the ones that define the building.  $\square$

### § 3. LOOP GROUPS

Let  $LG, LG_{\mathbf{C}}$  denote the free loop spaces. Let  $G_{\mathbf{C}}$  denote the group of loops which are restrictions of regular maps  $\mathbf{C}^* \rightarrow G_{\mathbf{C}}$ , and let  $L_{\text{alg}}G = L_{\text{alg}}G_{\mathbf{C}} \cap LG$ . Thus if we fix an embedding  $G_{\mathbf{C}} \subset GL(n, \mathbf{C})$ ,  $L_{\text{alg}}G$  consists of the loops  $f$  in  $LG$  admitting a finite Laurent expansion  $f(z) = \sum_{k=-m}^m A_k z^k$ , whereas  $L_{\text{alg}}G_{\mathbf{C}}$  consists of the loops  $f$  in  $LG_{\mathbf{C}}$  such that both  $f$  and  $f^{-1}$  admit finite Laurent expansions. We will also write  $\tilde{G}_{\mathbf{C}}$  for  $L_{\text{alg}}G_{\mathbf{C}}$ . In fact  $\tilde{G}_{\mathbf{C}}$  is the group of points over  $\mathbf{C}[z, z^{-1}]$  of the algebraic group  $G_{\mathbf{C}}$ . Its Lie algebra is the loop algebra  $\tilde{g}_{\mathbf{C}}$  of regular maps  $\mathbf{C}^* \rightarrow g_{\mathbf{C}}$ . The integer  $m$  in the above Laurent expansion defines a filtration of  $\tilde{G}_{\mathbf{C}}$  by finite dimensional subspaces; we give  $\tilde{G}_{\mathbf{C}}$  the corresponding weak topology.

Let  $P$  denote the subgroup of  $\tilde{G}_{\mathbf{C}}$  consisting of regular maps  $\mathbf{C} \rightarrow G_{\mathbf{C}}$  (i.e. maps with nonnegative Laurent expansion, or  $G_{\mathbf{C}[z]}$ ), and let  $\tilde{B}$  denote the Iwahori subgroup:  $\{f \in P : f(0) \in B^{-}\}$ . Finally, let  $\tilde{N} = L_{\text{alg}}N_{\mathbf{C}}$ , and recall that  $\tilde{W}$  can be regarded as a "subgroup" of  $\tilde{G}_{\mathbf{C}}$ , since  $R \leq \text{Hom}(S^1, T) \leq L_{\text{alg}}T$ . More precisely, we have  $\tilde{N}/T_{\mathbf{C}} = \hat{W}$ , and  $\tilde{W} \subset \hat{W}$ .

The *affine root system*  $\Phi$  is the set  $\mathbf{Z} \times \Phi$ . It can be thought of as a set of affine linear functionals on  $t$ , but for our purposes it is just a device for encoding combinatorial information about the affine Weyl group and  $\tilde{G}_C$ . In particular, to each  $(n, \alpha) \in \Phi$  we associate a root subalgebra  $X_{n, \alpha}$  of  $\tilde{g}_C$  consisting of the regular maps  $\mathbf{C}^* \rightarrow X_\alpha$  homogeneous of degree  $n$ . These subalgebras are one-dimensional, and are precisely the nontrivial eigenspaces of the following  $T^{l+1}$  action: The constant loops  $T^l$  act in the obvious way, and the extra  $S^1$  factor acts by rotating the loops. We also have root subgroups  $U_{(n, \alpha)} = \exp X_{n, \alpha} \leq \tilde{G}_C$ . One can easily check that  $\tilde{W}$  (acting by left conjugation) permutes the root subgroups. The resulting action of  $\tilde{W}$  on  $\tilde{\Phi}$  is given by  $(w\lambda) \cdot (n, \alpha) = (n + \alpha(\lambda), w\alpha)$  for  $\lambda \in \text{hom}(S^1, T)$ ,  $w \in W$ . The various additional structures associated with ordinary root systems can be defined here as well. The positive roots  $\tilde{\Phi}^+$  are the  $(n, \alpha)$  with  $n \geq 1$  or  $n = 0$  and  $\alpha < 0$  (note these correspond to the Iwahori subgroup  $\tilde{B}$ ); the remaining roots are negative. As in the finite case, the length of an element  $\sigma$  in  $\tilde{W}$  is equal to the number of positive roots taken to negative roots by  $\sigma$  (in particular this latter number is finite, as is clear anyway from the above formula for the  $\tilde{W}$  action). The simple affine roots are defined as the set of elements of  $\tilde{\Phi}^+$  which are indecomposable with respect to addition:  $(m, \alpha) + (n, \beta) = (m+n, \alpha+\beta)$  (if  $\alpha+\beta$  is a root). Hence the simple roots are  $(0, -\alpha), \dots, (0, -\alpha_l)$  and  $(1, \alpha_0)$ .

To each root  $(n, \alpha)$ , we can also associate a "little  $SL_2$ " subgroup generated by  $U_{n, \alpha}$  and  $U_{-n, -\alpha}$ . In particular  $\tilde{G}_{C, i}$  is the subgroup corresponding to the  $i$ th simple affine root,  $0 \leq i \leq l$ . Thus  $\tilde{G}_{C, i} = G_{C, i}$  if  $i \neq 0$ , and  $\tilde{G}_{C, 0}$  corresponds to  $(1, \alpha_0)$ . For example, if  $G = SU(2)$ ,  $\tilde{G}_{C, 0}$  is the subgroup of matrices  $\begin{pmatrix} a & bz \\ cz^{-1} & d \end{pmatrix}$  with  $ad - bc = 1$ . We let  $\tilde{G}_i = \tilde{G}_{C, i} \cap LG$ . Again  $\tilde{G}_i = G_i$  if  $i \neq 0$ . Note that for all  $i$ , evaluation at  $z = 1$  gives an isomorphism  $\tilde{G}_i \xrightarrow{\cong} G_i \cong SU(2)$ .

(3.1) THEOREM. Assume  $G$  is simply-connected. Then  $(\tilde{G}_C, \tilde{B}, \tilde{N}, \tilde{S})$  is a topological Tits system satisfying the four axioms of § 2.

*Proof.* That  $(\tilde{G}_C, \tilde{B}, \tilde{N}, \tilde{S})$  is a Tits system in the ordinary sense is essentially due to Iwahori and Matsumoto [16]. (They work over a complete local field  $K$ ; here we take  $K$  to be the field of infinite Laurent series bounded below. It is not hard to get from the Chevalley group  $G_K$  to  $G_{\mathbf{C}[z, z^{-1}]} = \tilde{G}_C$ .) See also Kac and Peterson [17].

Clearly  $\tilde{B}$  and  $\tilde{N}$  are closed subgroups and  $\tilde{W}$  is discrete. For Axiom (2.11) we need to show that if  $\tilde{W}$  is an irreducible affine Weyl group,

and  $I$  is a proper subset of  $\tilde{S}$ , then  $\tilde{W}_I$  is finite. This is obvious since the elements of  $I$  have a common fixed point (i.e. the intersection of the corresponding reflection hyperplanes is nonempty). In Axiom (2.12) we take  $A_s = \tilde{G}_s$ . We have  $\tilde{G}_s \tilde{B} = \tilde{G}_{c,s} \tilde{B} = \tilde{B}$   $U_s \tilde{B} = P_s$ . In particular  $P_s / \tilde{B} = \tilde{G}_s / (\tilde{G}_s \cap \tilde{B}) \cong SU(2)/T = \mathbb{C}P^1$ , which also proves Axioms (2.20) and (2.21).  $\square$

(3.2) COROLLARY.  $\Omega_{alg} G$  is a CW-complex with cells of even dimension, indexed by  $\text{Hom}(S^1, T)$ . The Poincaré series for its integral homology is  $\sum_{\lambda \in \text{Hom}(S^1, T)} t^{2\bar{l}(\lambda)}$ , where  $\bar{l}(\lambda)$  is the minimal length accruing in  $\lambda W$ . Identifying  $\text{Hom}(S^1, T)$  with  $\tilde{W}^S$ , the closure relations on the cells are given by the Bruhat order on  $\tilde{W}^S$ .  $\square$

*Remark.* An explicit formula for  $\bar{l}(\lambda)$  is given in [16], Prop. 1.25:  $\bar{l}(\lambda) = (\sum_{\alpha > 0} |\alpha(\lambda)|) - |\{\alpha > 0 : \alpha(\lambda) > 0\}|$ .

We will also need the “Iwasawa decomposition” (see [17], [27], [29]):

(3.3) THEOREM.  $\tilde{G}_C = \Omega_{alg} G \times P$ .  $\square$

*Remark.* Note that (3.3) shows that the associated building, which we will be denoted simply by  $\mathcal{B}_G$ , is a quotient of  $L_{alg} G/T \times \Delta$ . The equivalence relation is then  $(f_1 T, X) \sim (f_2 T, X)$  if  $X \in \dot{\Delta}_I$  and  $f_1 = f_2 \bmod LG \cap P_I$ .

#### § 4. QUILLEN'S THEOREM FOR LOOP GROUPS

In this section we will give Quillen's proof of the following theorem.

(4.1) THEOREM. Let  $G$  be a compact Lie group. Then the inclusion  $\Omega_{alg} G \rightarrow \Omega G$  is a homotopy equivalence.

If  $G$  is simply connected, let  $\mathcal{B}_G$  denote the topological building associated to the algebraic loop group  $L_{alg} G_C$  as in § 2.

(4.2) THEOREM (Quillen).  $\Omega_{alg} G$  acts freely on  $\mathcal{B}_G$ , with orbit space  $G$ .

*Proof of (4.1).* It is easy to reduce to the case when  $G$  is simply connected. Since  $B_G$  is contractible by Theorem 2.16, we conclude at once from Theorem (4.2) that  $\Omega_{alg} G \rightarrow \Omega G$  is a weak equivalence. Since both spaces have the homotopy type of a CW-complex, the map is in fact a homotopy equivalence.  $\square$