

§3. Algebraic lemmas

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where $r = (m-1)/2$ and $\varepsilon(i) = (-1)^{i+1}$. It is easy to show that under a different choice of natural bases and bases $h_0, h_1, \dots, h_m$ the element d is replaced by $\pm gq\bar{q}d$ with $g \in G, q \in Q \setminus 0$. Thus the set $\{\pm gq\bar{q}d \mid g \in G, q \in Q \setminus 0\} \subset Q$ does not depend on the choice of bases. It also does not depend on the choice of triangulation in M . It is this set which is $\omega(M)$.

An explicit formula established in [4] enables us to calculate $\omega(M)$ in terms of the orders of $\mathbf{Z}[G]$ -modules $H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; \mathbf{Z})$, $H_*(\tilde{M}) = H_*(\tilde{M}; \mathbf{Z})$ and related modules. (The notion of the order of a module is recalled in Sec. 3.1.) Denote by J the image of the inclusion homomorphism $H_r(\partial\tilde{M}) \rightarrow H_r(\tilde{M})$ where $r = (m-1)/2$. Then up to multiples of type $q\bar{q}$ with $q \in Q \setminus 0$

$$(4) \quad \omega(M) = \text{ord}(\text{Tors}_{\mathbf{Z}[G]} H_r(M, \partial M)) (\text{ord } J)^{\varepsilon(r)} \prod_{i=0}^{r-1} [\text{ord } H_i(\partial M)]^{\varepsilon(i)}$$

(see [4, Theorem 5.1.1]). Note that the equalities $Q \otimes_{\mathbf{Z}[G]} H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; Q) = 0$ imply that $H_*(\partial\tilde{M})$ and J are torsion $\mathbf{Z}[G]$ -modules. Therefore $\text{ord } H_i(\partial\tilde{M})$ and $\text{ord } J$ are non-zero elements of $\mathbf{Z}[G]$.

We shall apply formula (4) in the case where M is the exterior of an n -component link $K \subset S^m$ with odd m . The condition $H_*(\partial M; Q) = 0$ is always fulfilled in this case. Here the field Q is canonically identified with the field of rational functions of n variables $Q_n = Q(t_1, \dots, t_n)$. Thus $\omega(M) \subset Q_n$. If $m \geq 5$ then (4) implies that

$$\Delta(K)(t_1, \dots, t_n) \cdot \prod_{i=1}^n (t_i - 1) \subset \omega(M).$$

If $m = 3$ then there exists a unique subset $\alpha = \alpha(K)$ of the set $\{1, 2, \dots, n\}$ such that

$$\Delta_{u(K)}(K)(t_1, \dots, t_n) \cdot \prod_{i \in \alpha} (t_i - 1) \subset \omega(M).$$

For proofs and details consult [4, § 5].

§ 3. ALGEBRAIC LEMMAS

3.1. PRELIMINARY DEFINITIONS. For a finitely generated module H over a (commutative) domain R we denote by $\text{rk}_R H$ or, briefly, by $\text{rk } H$ the integer $\dim_Q(Q \otimes_R H)$ where $Q = Q(R)$ denotes the field of fractions of R . For a R -linear homomorphism $f: H \rightarrow H'$ we put $\text{rk } f = \text{rk}_R f(H)$. Note that if $\bar{R}$ is the localization of R at some multiplicative system then $Q(\bar{R}) = Q(R)$ and therefore the (exact) functor $(H \mapsto \bar{R} \otimes_R H, f \mapsto \text{id}_{\bar{R}} \otimes f)$

preserves the ranks of modules and homomorphisms. If H, H' are finitely generated free R -modules and if A is the matrix of a R -homomorphism $H \rightarrow H'$ with respect to some bases then $\text{rk } f = \text{rk } A$ where $\text{rk } A$ is the maximal integer r such that some $r \times r$ -minor of A is non-zero.

If R is a unique factorization domain with 1 and if A is a matrix with $n < \infty$ columns and possibly infinite number of rows then $\Delta_i(A)$ denotes the greatest common divisor of the $(n-i+1) \times (n-i+1)$ -minors of A . Here $i = 1, 2, \dots$ and $\Delta_i(A)$ is an element of R defined up to a unit multiple. If H is a finitely generated module over R and A is a presentation matrix of H then $\Delta_i(A)$ depends only on H and i ; one defines $\Delta_i(H) = \Delta_i(A)$. Clearly $\Delta_i(H) = 0$ for $i \leq \text{rg } H = n - \text{rg } A$ and $\Delta_i(H) \neq 0$ for $i > \text{rg } H$. The invariant $\Delta_1(H)$ is denoted also by $\text{ord } H$; it is called the order of H . It is clear that $\text{ord } H \neq 0$ iff $H = \text{Tors}_R H$. For proofs and further information see [1].

Recall, finally, that a local ring is a domain K which has a unique maximal (proper) ideal. The quotient of K by this ideal is a field which we shall call "the field associated to K ".

3.2. LEMMA. *Let R, R' be (commutative) domains with 1 and let $\varphi: R \rightarrow R'$ be a ring homomorphism. Let $C = (\cdots \rightarrow C_{i+1} \rightarrow C_i \rightarrow \cdots)$ be a finitely generated free chain complex over R and let C' be the chain R' -complex $R' \otimes_R C$. Then: (i) $\text{rk}_{R'} H_i(C') \geq \text{rk}_R H_i(C)$ and $\text{rk } \partial'_i \leq \text{rk } \partial_i$ for all i where ∂_i, ∂'_i are the boundary homomorphisms $C_{i+1} \rightarrow C_i, C'_{i+1} \rightarrow C'_i$; (ii) if $\text{rk } H_i(C') = \text{rk } H_i(C)$ for some i then $\text{rk } \partial'_j = \text{rk } \partial_j$ for $j = i, i+1$; (iii) if R, R' are unique factorization Noetherian domains and if $\text{rk } H_i(C') = \text{rk } H_i(C)$ then $\varphi(\text{ord}(\text{Tors}_R H_i(C)))$ divides $\text{ord}(\text{Tors}_{R'} H_i(C'))$.*

Proof. Let $n = \text{rk } C_i$. Let $A = (a_{p,q})$, $1 \leq q \leq n$, $1 \leq p$, be the matrix of ∂_i with respect to some bases in C_i, C_{i+1} . Then $A' = (\varphi(a_{p,q}))$ is the matrix of ∂'_i with respect to the induced bases in C'_i, C'_{i+1} . It is evident that $\text{rk } \partial'_i = \text{rk } A' \leq \text{rk } A = \text{rk } \partial_i$. Therefore

$$\text{rk } H_i(C') = n - \text{rk } \partial'_i - \text{rk } \partial'_{i+1} \geq n - \text{rk } \partial_i - \text{rk } \partial_{i+1} = \text{rk } H_i(C).$$

These inequalities imply (i) and (ii).

Put $r = n - \text{rk } A + 1$ and denote the R -module $C_i/\text{Im } \partial_i$ by J . Since A is a presentation matrix of J we have $\text{ord}(\text{Tors}_R J) = \Delta_r(A)$ (see [1, p. 31]). From the exact sequence $0 \rightarrow H_i(C) \rightarrow J \rightarrow C_{i-1}$ we obtain that $\text{Tors } J = \text{Tors } H_i(C)$. Thus $\text{ord}(\text{Tors } H_i(C)) = \Delta_r(A)$. Analogously $\text{ord}(\text{Tors } H_i(C')) = \Delta_{r'}(A')$ where $r' = n - \text{rk } A' + 1$. If $\text{rk } H_i(C) = \text{rk } H_i(C')$ then $\text{rk } A = \text{rk } A'$ and therefore $r = r'$. It is evident that $\varphi(\Delta_j(A))$ divides $\Delta_j(A')$ for all j . This implies (iii).

3.3. LEMMA. Let R be a local ring and F be the associated field. Let $f: C_1 \rightarrow C_0$ be a R -homomorphism of finitely generated free R -modules and let $\bar{f}: F \otimes_R C_1 \rightarrow F \otimes_R C_0$ be the induced F -homomorphism. If $\text{rk } f = \text{rk } \bar{f}$ then with respect to some bases in C_1, C_0 the homomorphism f is presented by the matrix $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ where E is the unit matrix of order $\text{rk } f$.

Proof. Since F is a field we can choose bases d_0, d_1 respectively in $F \otimes_R C_0, F \otimes_R C_1$ so that the matrix of $\bar{f}$ regarding these bases has the form $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$. Let $\mathcal{D}_i$ be a lifting of d_i to $C_i, i = 1, 2$. Here $\mathcal{D}_i$ is a sequence of $\text{rg } C_i$ elements of C_i . In view of Nakayama's lemma $\mathcal{D}_i$ generate C_i . This implies that $\mathcal{D}_i$ generates the $(\text{rg } C_i)$ -dimensional vector space $Q(R) \otimes_R C_i$ over the field $Q(R)$. Therefore, the elements of the sequence $\mathcal{D}_i$ are linearly independent over $Q(R)$ and, hence, over R . Thus $\mathcal{D}_i$ is a basis of C_i for $i = 0, 1$. The matrix of f with respect to bases $\mathcal{D}_0, \mathcal{D}_1$ has the form $\begin{bmatrix} E+U & Z \\ X & Y \end{bmatrix}$ where U, X, Y, Z are matrices over the maximal ideal u of R . Note that $\det(E+U) = 1 \pmod{u}$. Since all elements of $R \setminus U$ are invertible in R the square matrix $E+U$ is invertible over R . Therefore we can choose bases in C_0, C_1 so that the corresponding matrix of f equals $\begin{bmatrix} E & 0 \\ 0 & Y' \end{bmatrix}$. Since $\text{rk } f = \text{rk } \bar{f} = \text{rk } E, Y' = 0$.

3.4. LEMMA. Let R be a local ring and F be the associated field. Let $C = (\cdots \rightarrow C_{i+1} \rightarrow C_i \rightarrow \cdots)$ be a finitely generated free chain complex over R . Let C' be the chain F -complex $F \otimes_R C$. Let ∂_i, ∂'_i be the boundary homomorphisms $C_{i+1} \rightarrow C_i, C'_{i+1} \rightarrow C'_i$. If $\text{rk}_R H_i(C) = \text{rk}_F H_i(C')$ for some i then: $H_i(C), \text{Im } \partial_{i+1}, \text{Im } \partial_i$ are free R -modules and $C_i = \text{Im } \partial_{i+1} \oplus H_i(C) \oplus \text{Im } \partial_i$; the projection $C \rightarrow C'$ induces F -isomorphisms $F \otimes_R H_i(C) \rightarrow H_i(C'), F \otimes_R \text{Im } \partial_j \rightarrow \text{Im } \partial'_j$ with $j = i, i+1$.

This Lemma directly follows from Lemmas 3.2 (ii) and 3.3.

§ 4. PROOF OF THEOREMS 1 AND 2

4.1. PROOF OF THEOREM 1. Denote by Q_n the fraction field of the ring $\Lambda_n = \mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$. Denote by Q_n^0 the subring of Q_n which consists of rational functions fg^{-1} with $f, g \in \Lambda_n$ and $g \notin (t_n - 1)\Lambda_n$ (so that