# 2. Construction of $O(\tilde{X})$ from O(X) for Stein spaces X

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According to a theorem of Oka [12], the normalization sheaf $\tilde{\mathcal{O}}$ of weakly holomorphic functions on a complex space $(X, \mathcal{O})$ is coherent. Consequently, there is a canonical topology making $\tilde{\mathcal{O}}$ a Fréchet sheaf; the global weakly holomorphic functions $\tilde{\mathcal{O}}(X)$ will always carry this topology. Since the holomorphic functions $\mathcal{O}(\tilde{X})$ on the normalization $\tilde{X}$ of $X$ are topologically isomorphic to $\tilde{\mathcal{O}}(X)$ [8, 8.3], the question posed in the introduction can now be answered.

Main theorem. *For an irreducible Stein space $X$, the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ is dense in $\tilde{\mathcal{O}}(X)$.*

*Proof.* Let $\pi: \tilde{X} \to X$ be the normalization of $X$ and put $A := \widetilde{\mathcal{O}(X)}$. Since $\pi$ is proper, $\tilde{X}$ is $\mathcal{O}(X)$-convex and therefore $\bar{A}$-convex. Note that Corollary 1 implies $A \subset \tilde{\mathcal{O}}(X)$ and that $\bar{A}$ is the closure of $A$ with respect to the canonical topology in $\tilde{\mathcal{O}}(X)$.

Consider the equivalence relation $R$ on $\tilde{X}$ defined by $\bar{A}$, i.e. $(x, y) \in R$ iff for every $f \in \bar{A}$, $f(x) = f(y)$. Rossi's theorem [13] ensures that the topological quotient $Y := \tilde{X}/R$ can be given the complex structure of a Stein space such that the projection $p: \tilde{X} \to Y$ is holomorphic and proper and the map $p^*: \mathcal{O}(Y) \to \mathcal{O}(\tilde{X})$, $f \mapsto f \circ p$, induces an isomorphism $\mathcal{O}(Y) \cong \bar{A}$.

It suffices to show that every $f \in \mathcal{O}(\tilde{X})$ can be factorized through a holomorphic function on $Y$, meaning that an $F \in \mathcal{O}(Y)$ exists with $F \circ p = f$. This will be accomplished by first factorizing $f \in \mathcal{O}(\tilde{X})$ through a continuous function $F$ on $Y$ and then proving that $F$ is actually holomorphic. The existence of such a continuous factor $F$ for $f$ is equivalent to demonstrating that every $f \in \mathcal{O}(\tilde{X})$ is constant on the fibers of $p$. The validity of this geometric statement will be shown now using commutative algebra.

$\mathcal{O}(\tilde{X})$ is almost integral over $\mathcal{O}(X)$ (see § 1), and hence over the localization $S_x^{-1}A$ of $A$ with respect to $S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\}$ for every $x \in X$. Moreover,

$$S_x^{-1}A = \widetilde{S_x^{-1}\mathcal{O}(X)}$$

holds [3, V, 1.5, Corollary 1]. The localization $S_x^{-1}\mathcal{O}(X) = \mathcal{O}(X)_{m(x)}$ of the Stein algebra $\mathcal{O}(X)$ at the maximal ideal $m(x) := \{f \in \mathcal{O}(X) : f(x) = 0\}$ is noetherian — even more, it's excellent [2, p. 35]. According to a theorem of Mori-Nagata, the integral closure of a noetherian integral domain is completely normal [7, 4.3, 3.6], implying

$$(*) \qquad \mathcal{O}(\tilde{X}) \subset \bigcap_{x \in X} S_x^{-1} A \, .$$

For $f \in \mathcal{O}(\tilde{X})$, $a \in \tilde{X}$ and $b \in p^{-1}(p(a))$, it is now possible to conclude that $f(a) = f(b)$ is true. Let $x := \pi(a)$. Due to $(*)$, functions $g \in S_x$ and $h \in A$ exist with $f = h/g \circ \pi$. Since $a$ and $b$ are equivalent with respect to the equivalence relation $R$, $f(a) = f(b)$ follows, and a continuous function $F: Y \to \mathbf{C}$ exists with $F \circ p = f$.

Since the Stein complex structure on $Y$ is not in general the canonical ringed quotient structure, it is still necessary to verify that $F$ is holomorphic in order to prove the density of $A$ in $\mathcal{O}(\tilde{X})$. To that end, let $H \in \mathcal{O}(Y)$ and $G \in \mathcal{O}(Y)$ have the property that $H \circ p = h$ and $G \circ p = g \circ \pi$. Such functions exist because $p^*(\mathcal{O}(Y)) = \bar{A}$ holds. Then $F = H/G$ follows, and the germ $F_{p(a)}$ is the germ of a holomorphic function at $p(a)$, since the germ $G_{p(a)}$ of $G$ at $p(a)$ is a unit. The surjectivity of $p$ implies that $F$ is holomorphic on $Y$, completing the proof of the theorem.

Note that the topology induced by $\mathcal{O}(\tilde{X})$ on any subalgebra $A$ of $\mathcal{O}(\tilde{X})$ is the metrizable topology of uniform convergence on compact subsets of $X$. Because the closure $\bar{A}$ of $A$ in $\mathcal{O}(\tilde{X})$ is its completion, $\bar{A}$ can be obtained without referring directly to $\mathcal{O}(\tilde{X})$. Thus the Main Theorem can be stated as follows:

*If $\tilde{X}$ denotes the normalization of an irreducible Stein space $X$, then $\mathcal{O}(\tilde{X})$ is the completion of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$.*

## 3. Applications

In this section $X$ will denote an irreducible Stein space with normalization $\pi: \tilde{X} \to X$, $\widetilde{\mathcal{O}(X)}$ will be the integral closure of the holomorphic functions $\mathcal{O}(X)$ on $X$, $\tilde{\mathcal{O}}(X)$ the Fréchet algebra of weakly holomorphic functions on $X$ (or equivalently, the Fréchet algebra of holomorphic functions $\mathcal{O}(\tilde{X})$ on $\tilde{X}$), and

$$S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\} \quad \text{for} \quad x \in X \, .$$

Although the example given in the first section shows that the algebras $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ are not always equal, the inclusion $(*)$ in the proof of the Main Theorem implies that they are locally equal in the following sense.

THEOREM 2. *For every $x \in X$, the localizations of $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ with respect to $S_x$ coincide.*