§7. Atiyah-Ward ansatzes, summing 't Hooft solutions and elsenstein series

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

§ 7. Atiyah-Ward ansatzes, summing 't Hooft solutions and Eisenstein series

In this section we shall derive some explicit formulae for monopoles on handlebodies, using the complex geometry of their twistor spaces. A detailed study of the moduli spaces of monopoles on a solid torus has been made in Braam-Hurtubise [11].

From the description of Z as $P(S_+)$, it follows that on Z there exists a *tautological line bundle* L, which upon restriction to the fibre over $x \in X$, equals the negative Hopf bundle on $P(S_{+,x})$. It turns out that L is naturally holomorphic, and to tie in with the $(\mathbf{CP}^3, S^4)$ case we shall denote the $(-q)$-th power of L by $\mathcal{O}(q)$.

If $F \to \mathbf{CP}^3$ is an instanton bundle on the twistor space of S^4 then *Atiyah-Ward ansatzes*, that is an explicit formula for the instanton on S^4, arise from a suitable description of F as holomorphic bundle. Let s be a section of $F \otimes \mathcal{O}(q) = F(q)$. Generically s will be nonzero away from a *complex curve* $C_s \subset Z$ and give rise to an extension class $e_s \in H^1(Z - C_s, \mathcal{O}(-2q))$. Elements of such sheaf cohomology groups correspond to solutions ϕ_s of linear p.d.e. on open sets of S^4: this is the celebrated *Penrose correspondence*. Explicit formulas for the instanton, such as those of 't Hooft, can be constructed in terms of this ϕ_s. Every instanton on S^4 can theoretically be computed in this way. For background see Atiyah [1].

We shall see that on our manifolds $X = (S^4 - \Lambda)/\Gamma$, for $\Gamma \neq \{e\}$, the situation is rather different, but that nevertheless in some cases explicit constructions can be made again. As before attention will only be paid to $\tilde{S}^1$-invariant instantons, i.e. monopoles. In those cases which we treat in detail, it will appear that we are essentially summing together a monopole, much in the same way as automorphic forms are constructed by summing kernels. It is however quite remarkable that "summing" of solutions is possible for the non-linear anti-self-duality equations, and may be these summation procedures are best thought of as a kind of Backlund transformations.

Recall from § 2 and § 3, that X comes with a natural conformal structure, and that X can be given a metric in the conformal class with constant scalar curvature R_X. We proved that the majority of X's give rise to negative R_X. Assume a spin structure on X has been fixed, then the line bundle $\mathcal{O}(q)$ above is well defined.

PROPOSITION 7.1. *If* $R_X < 0$, *then no monopole on* X *arises from an Atiyah-Ward construction, since* $H^0(Z, F(q)) = 0$ *for all* $q \in \mathbf{Z}\backslash\{0\}$.

Proof. For $q < 0$ any section would vanish on the fibres $\pi^{-1}(x)$, and hence be zero; this is independent of the sign of R_X. For $q > 0$, we know from Hitchin [18], that elements of $H^0(Z, F(q))$ are in one-one correspondence with solutions of the twistor equation on X with coefficients in $E = P \times_{SU(2)} \mathbf{C}^2$:

$$\bar{D}_q s = 0$$

$$\bar{D}_q = \mathscr{P} \circ \nabla_A : \Gamma(S^q(S_+) \otimes E) \to \Gamma(S^{q+1}(S_+) \otimes S_- \otimes E),$$

with S^q the q-th symmetric product, $\mathscr{P}: \Lambda^1 \otimes S^q(S_+) \to S^{q+1}(S_+) \otimes S_-$ the projection, and A the anti-self-dual $SU(2)$-connection on $E \to X$. For these equations we have a vanishing theorem of Weizenbock type in the case of negative scalar curvature, see Besse [8]. □

Hence attention here needs only be paid to the $R_X \geqslant 0$ manifolds, which were classified in theorem 3.1. But even here there is a very fundamental difference between the case $X = S^4$, i.e. $\Gamma = \{e\}$, and the cases of non-trivial Γ.

On $X = S^4$, $Z = \mathbf{CP}^3$, the dimensions of $H^0(Z, \mathcal{O}(q))$ (and also of the invariant part $H^0(Z, \mathcal{O}(q))^{S^1}$) increase with q. Tracing through the (equivariant) Riemann-Roch formula (as in Hitchin [19]), one learns that the increasing character is due to the fact that for the fixed point sets $S^+ = P_1^+$, $S^- = P_1^- \subset Z = \mathbf{CP}^3$ we have $\chi(S^\pm) > 0$. For $\Gamma \neq \{e\}$ these Euler characteristics satisfy $\chi(S^\pm) \leqslant 0$. This leads one to suspect that it may not always be possible to find sections of $F(q)$, which would be needed to obtain Atiyah-Ward ansatzes in general.

After all these negative remarks, let us proceed to show that, at least in some cases, the construction works satisfactorily. To simplify things even further, we shall assume that X is a manifold with $R_X > 0$; by theorem 3.1, X arises from a Schottky group. Consider on X the *conformally invariant Laplacian* D_0 acting on densities of conformal weight 1, with values in densities of weight 3, which equals

$$D_0 = d^*d + \frac{1}{6} \cdot R_X .$$

Since $R_X > 0$, we get $\ker D_0 = 0$, and hence unique fundamental solutions ϕ_x exist satisfying

$$D_0 \cdot \phi_x = \delta_x \qquad x \in X .$$

Through the twistor correspondence (see Atiyah [3], [1], and Hitchin [18]) ϕ_x corresponds to a *cohomology class*:

$$\varphi_x \in H^1(Z - \pi^{-1}(x), \mathcal{O}(-2)),$$

and hence ϕ_x gives rise to a vector bundle F on $Z - \pi^{-1}(x)$, which is an extension:

$$0 \to \mathcal{O}(-1) \to F \to \mathcal{O}(1) \to 0 .$$

In fact one can show (Atiyah [3]) that the bundle F extends to a bundle F on Z, such that $F(1)$ has a holomorphic section vanishing precisely on $\pi^{-1}(x)$. The maximum principle applied to D_0 ensures that $\phi_x(y) > 0$, for all $y \in X$, and this implies that F is trivial on the real lines $\pi^{-1}(x)$. Since ϕ_x is real, F gets a real structure. Thus F is an instanton bundle.

To get a monopole rather than just an instanton we have to assume $x \in S_1$, the fixed surface in X. The weight m_1 of a monopole constructed in this way equals 1, because the Hopf bundle $\mathcal{O}(1)$ is of weight 1. The charge also equals 1.

Obviously the process can be generalized by using a positive linear combination of k fundamental solutions:

$$\varphi = \Sigma \lambda_j \varphi_{x_j} \qquad \lambda_j > 0 , \quad j = 1, \ldots, k ,$$

which is called an *'t Hooft potential.* If the x_j lie in $S_1 \subset X$, then the 't Hooft potential will be invariant, and it follows that we have created a monopole of mass 1 and charge k. All positive scalar multiples of ϕ give the same instanton, so the number of parameters in the solutions is $3k - 1$: we have 2 for every $x_j \in S_1$, and 1 for every λ_j. These solutions therefore don't give an open set in the $4k - \frac{1}{2} \cdot \chi(S)$ dimensional moduli space.

We proceed to identify these potentials ϕ. In the course of this, explicit formulas for the connection A will also be given. Besides, a slight generalization of the Atiyah-Ward construction will emerge.

Pulling back ϕ_x to $S^4 - \Lambda$, under the quotient map, one gets a generalized function $\tilde{\phi}_x$ on $S^4 - \Lambda$ satisfying:

$$D_0 \tilde{\varphi}_x = \sum_{\gamma \in \Gamma} \delta_{\gamma y}$$

with $y \in S^4 - \Lambda$ mapping to x. Of course the next step is to try to reverse this and to put:

7.1 $$\tilde{\varphi}_x = \sum_{\gamma\in\Gamma} \psi_{\gamma y}$$

where ψ_y is a fundamental solution on S^4 of D_0 at y. In the flat metric on $\mathbf{R}^4 \subset S^4$, fundamental solutions are equal to:

7.2 $$\psi_y(r) = (2\pi\|y-r\|)^{-2}\,.$$

Since the flat metric is not Γ-invariant, conformal weight factors will occur in 7.1. It is easier to see what happens if one uses the Γ-invariant metric on $H^3 \times S^1$:

$$t^{-2}(dx_1^2+dx_2^2+dt^2) + d\theta^2 \qquad (x_1, x_2, t, \theta) \in H^3 \times S^1$$

Under conformal rescaling, 7.2 transforms to the θ-independent *summation kernel of the Eisenstein series* on H^3 (compare Mandouvalos [25]):

$$E(y, h) = t/[(x_1-y_1)^2+(x_2-y_2)^2+t^2] \quad y \in \mathbf{R}^2 \subset S^2, \quad h = (x_1, x_2, t) \in H^3$$

Summing, we get for 7.1:

7.3 $$E_\Gamma(y, h) = \sum_{\gamma\in\Gamma} E(y, \gamma h)\,,$$

which is the *Eisenstein series* for Γ, see Mandouvalos [25]. As settled by Poincaré already, 7.3 is convergent if $\delta(\Gamma) < 1$, where $\delta(\Gamma)$ is the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ of Γ. The groups Γ for which this holds are the cyclic groups and classical Schottky groups (with their defining circles wide apart, compare Bers [7]). In passing by we note that $\delta(\Gamma) < 1$ implies that X is of positive type because the Eisenstein series is a strictly positive Green's function for $d^*d + \frac{1}{6}\cdot R_{sc}$: the maximum principle implies $R_{sc} > 0$.

To compute the gauge potentials, it is easiest to go back to the flat metric on $\mathbf{R}^4$. The more general potentials there look like

7.4 $$\phi(h, \theta) = \sum_{i=1}^{k} \lambda_i \cdot t^{-1} \cdot E_\Gamma(x_i, h)\,,$$

and the formulas of 't Hooft give for the connection (see Atiyah-Hitchin-Singer [5])

$$A = \sum_i P_+(-1/2\, d \log \varphi \Lambda e_i) \otimes e_i \in \Gamma(\mathbf{R}^4, \Lambda_+^2 \otimes \Lambda^1)\,,$$

with e_i an orthonormal, covariantly constant framing of $T^*\mathbf{R}^4$ and Λ_+^2 identified with $su(2)$. To see what this looks like, assume that Γ is cyclic, generated by $\begin{bmatrix} \lambda^{\frac{1}{2}} & \\ & \lambda^{-\frac{1}{2}} \end{bmatrix}$, $\lambda \in \mathbf{R}_{>0}$. Then

$$\varphi(r) = \sum_{n=1}^{\infty} \left[\sum_i \left\{ \frac{\lambda^n \lambda_i}{\| \lambda^n r - y_i \|^2} + \frac{\lambda^{-n} \lambda_i}{\| r - \lambda^{-n} y_i \|^2} \right\} \right] + \sum_i \frac{\lambda_i}{\| r - y_i \|^2} \tag{7.5}$$

$$\text{with } y_i \in \mathbf{R}^2 \subset S^2 \quad \text{and} \quad r \in S^4 \backslash \Lambda = \mathbf{R}^4 \backslash \{0\}.$$

So we see that for $\lambda \gg \lambda_i$ and $1 \leqslant \| r \|$, $\| y \| \leqslant \lambda$, the second term dominates strongly and the monopole will look much like a "grafted S^4-monopole". On making λ smaller, nearby nonlinear interaction makes the monopole look more complicated.

Finally we discuss a modification of this construction which supplies a few more solutions. Suppose we put $k = 1$ and consider the harmonic function:

$$\phi_\alpha(r) = \sum_{n \in \mathbf{Z}} \lambda^{(1+\alpha) \cdot n} \cdot \| \lambda^n r - y \|^{-2},$$

which converges for $-1 < \alpha < 1$. Then $\phi_\alpha(\lambda r) = \lambda^{-\alpha-1} \cdot \phi_\alpha(r)$, so the instanton is invariant. This results in a 3-parameter family of monopoles.

Now ϕ_α describes a fundamental solution of the Laplacian acting on sections of a flat real line bundle with monodromy λ^α along the non-trivial loop in H^3/Γ, so we have constructed a bundle F on twistor space, which is an extension of $L(1)$ by $L^*(-1)$, where L is a real flat line bundle in the Picard group of Z with monodromy $\lambda^{\frac{1}{2} \cdot \alpha}$.

The same procedure can be used for Schottky groups Γ of genus g, by twisting the sum with a character $\Gamma \to \mathbf{R}_{>0}$ close to 1. This gives a $3k - \frac{1}{2} \cdot \chi(S)$ parameter family of monopoles. This too doesn't give an open set in the moduli spaces and it appears that the construction of the general solution is not yet clear, even in these simple cases.

Possibly this can be remedied by going over to the next Atiyah-Ward ansatz, which exploits the self-dual Maxwell equations on X. Here the vanishing sets could be choosen to be elliptic curves corresponding to closed geodesics in M.