

Appendix 2. The Hopf fibering and mutually isoclinic planes

Objektyp: **Appendix**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

- (ii) not commutative, i.e., generally, $XY \neq YX$ (but see (4) (iv) below);
- (iii) not associative, i.e., generally, $(XY)W \neq X(YW)$ (but see (7) below).
- (4) The *real part* of $X \equiv (q_1, q_2)$ is $\text{Re } X = (\text{Re } q_1, 0) \equiv \text{Re } q_1$. X is said to be *real* if $X = \text{Re } X$; i.e., (q_1, q_2) is real iff q_1 is real and $q_2 = 0$.
- (i) $\text{Re}(X + Y) = \text{Re}(X) + \text{Re}(Y)$.
- (ii) $\text{Re}(XY) = \text{Re}(YX)$.
- (iii) $\text{Re}(CX) = 0$ for all X implies that $C = 0$.
- (iv) $CX = XC$ for all X iff C is real. In this case, $C = (c_1, 0)$, where $c_1 = \text{real}$, and $CX = (c_1q_1, c_1q_2) = XC$.
- (5) The *conjugate* of $X \equiv (q_1, q_2)$ is $X^* = (q_1^*, -q_2)$.
- (i) $(X + Y)^* = X^* + Y^*$,
- (ii) $(XY)^* = Y^*X^*$.
- (iii) $X^* = X$ iff X is real.
- (6) The *norm* of X is the non-negative real number $N(X) \equiv XX^*$, which is also equal to X^*X . The *length* of X is the non-negative real number $|X| \equiv N(X)^{1/2} = (XX^*)^{1/2}$.
- (i) $N(X) = 0$ iff $X = 0$.
- (ii) If $X \neq 0$, then $X^{-1} \equiv X^*/N(X)$ is a right and left inverse of X .
- (iii) $N(XY) = N(X)N(Y)$. It follows from this that $XY = 0$ iff $X = 0$ or $Y = 0$.
- (7) Though multiplication is generally non-associative,
- (i) $(XY)Y^* = X(Y Y^*)$.
- (ii) If $Y \neq 0$, then $(XY)Y^{-1} = X = Y^{-1}(YX)$.
- (iii) $\text{Re}((XY)W) = \text{Re}(X(YW))$.

APPENDIX 2. THE HOPF FIBERING AND MUTUALLY ISOCLINIC PLANES

At the beginning of § 4, we described how H. Hopf obtained his fibering of S^{2n-1} by S^{n-1} over S^n , $n = 2, 4$, or 8 , by intersecting the unit sphere S^{2n-1} in $R^{2n} = Q_n \times Q_n$ with the Q_n -lines $Y = CX$ and $X = 0$. In Theorem 5.2, we proved that the Hopf fibering and maximal set of mutually isoclinic n -planes in R^{2n} are equivalent concepts. Here we prove, directly, the

THEOREM A2.1. *The set of Q_n -lines $\{Y = CX, X = 0\}$ in $Q_n \times Q_n$, when viewed as n -planes in R^{2n} , are mutually isoclinic n -planes.*

Proof. We shall prove the theorem for the case $n = 8$ only. The proof for the cases $n = 2, 4$ follows the same line and is simpler.

Some preliminaries are necessary. Suppose that under the identification of $Q_8 \times Q_8$ with R^{16} as in Theorem 5.1, the elements $(X, Y), (X', Y')$ of $Q_8 \times Q_8$ become the vectors $(X, Y), (X', Y')$ in R^{16} with respectively the components $(x_1, \dots, x_{16}), (x'_1, \dots, x'_{16})$. Then it can easily be verified that the inner product of the two vectors (X, Y) and (X', Y') is

$$\langle (X, Y), (X', Y') \rangle \equiv \sum_{i=1}^{16} x_i x'_i = \operatorname{Re} (XX'^* + YY'^*).$$

It follows from this that the length of the vector (X, Y) is

$$|(X, Y)| = \langle (X, Y), (X, Y) \rangle^{1/2} = (XX^* + YY^*)^{1/2},$$

and that the two vectors (X, Y) and (X', Y') are orthogonal if and only if $\operatorname{Re} (XX'^* + YY'^*) = 0$.

We can now prove our theorem by showing that in R^{16} , the 8-plane $\mathbf{A}: Y = AX$ is isoclinic with the 8-planes $\mathbf{B}: Y = BX$ and $\mathbf{O}^\perp: X = 0$.

Let $(T, BT) \in \mathbf{B}$ be the projection of any nonzero vector $(X, AX) \in \mathbf{A}$ on \mathbf{B} . Then the vector $(X - T, AX - BT)$ is orthogonal to \mathbf{B} , i.e., it is orthogonal to all the vectors $(W, BW) \in \mathbf{B}$, where W is an arbitrary Cayley number. Therefore,

$$(A.1) \quad \operatorname{Re} \{(X - T)W^* + (AX - BT)(BW)^*\} = 0 \quad \text{for all } W \in Q_8.$$

Since, by (4) (ii) and (7) (iii) in Appendix 1, the terms inside the brackets in $\operatorname{Re} \{ \quad \}$ are commutative and associative, the left-hand side of (A.1) is equal to

$$\begin{aligned} & \operatorname{Re} \{(X - T)W^* + [(AX - BT)W^*]B^*\} \\ &= \operatorname{Re} \{(X - T)W^* + [B^*(AX - BT)]W^*\} \\ &= \operatorname{Re} \{(X - T)W^* + [(B^*A)X - (B^*B)T]W^*\} \\ &= \operatorname{Re} \{[X - T + (B^*A)X - (B^*B)T]W^*\}. \end{aligned}$$

Therefore, by (4) (iii) in Appendix 1, condition (A.1) implies that

$$X - T + (B^*A)X - (B^*B)T = 0,$$

and hence

$$(A.2) \quad T = (1 + B^*A)X / (1 + B^*B).$$

Now, the squared length of the vector (X, AX) is

$$\begin{aligned} |(X, AX)|^2 &= XX^* + (AX)(AX)^* \\ &= XX^* + AA^*XX^*, \end{aligned}$$

i.e.,

$$(A.3) \quad |(X, AX)|^2 = (1 + A^*A)XX^*.$$

Similarly,

$$|(T, BT)|^2 = (1 + B^*B)TT^*.$$

But by (A.2),

$$\begin{aligned} TT^* &= (1 + B^*A)X[(1 + B^*A)X]^*/(1 + B^*B)^2 \\ &= (1 + B^*A)(1 + A^*B)XX^*/(1 + B^*B)^2. \end{aligned}$$

Therefore,

$$(A.4) \quad |(T, BT)|^2 = (1 + B^*A)(1 + A^*B)XX^*/(1 + B^*B).$$

Hence, it follows from (A.3) and (A.4) that the angle θ between the vector $(X, AX) \in \mathbf{A}$ and its projection on \mathbf{B} is given by

$$\cos^2\theta = \frac{|(T, BT)|^2}{|(X, AX)|^2} = \frac{(1 + A^*B)(1 + B^*A)}{(1 + A^*A)(1 + B^*B)},$$

which shows that the angle between any nonzero vector $(X, AX) \in \mathbf{A}$ and its projection on \mathbf{B} is independent of the choice of X ; that is, the 8-plane \mathbf{A} is isoclinic with the 8-plane \mathbf{B} .

Finally, to show that the 8-plane $\mathbf{A}: Y = AX$ is isoclinic with the 8-plane $\mathbf{O}^\perp: X = 0$, we need only observe that the projection of the nonzero vector $(X, AX) \in \mathbf{A}$ on \mathbf{O}^\perp is the vector (O, AX) , and

$$\frac{|(O, AX)|^2}{|(X, AX)|^2} = \frac{(AX)(AX)^*}{(1 + A^*A)XX^*} = \frac{AA^*}{1 + AA^*}$$

is independent of X .