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# AN ELEMENTARY PROOF OF THE STRUCTURE THEOREM FOR CONNECTED SOLVABLE AFFINE ALGEBRAIC GROUPS

by Dragomir Ž. ĐOKOVIĆ [1])

ABSTRACT

We give an elementary proof of the basic structure theorem for connected solvable affine algebraic groups $G$ over an algebraically closed field $k$. The main feature of our proof is that we first establish the important fact that the centralizer in $G$ of a semisimple element $s$ is connected. Then the main structure theorem follows easily. We also prove that such $s$ is contained in a maximal torus and that all maximal tori of $G$ are conjugate. The structure theorem for connected nilpotent affine groups is not needed in the proof; it is obtained at the end as a simple consequence of the main results. In our proof we avoid the use of quotients and Lie algebras of affine groups. On the other hand we use the Lie-Kolchin theorem, Chevalley's theorem, the existence and uniqueness of the Jordan decomposition, and some other elementary facts.

Let $k$ be an algebraically closed field. All algebraic groups will be defined over $k$ and are assumed to be affine. By $N \rtimes H$ we denote the semidirect product of affine algebraic groups where $N$ is a normal and $H$ a complementary subgroup. If $G$ is any affine algebraic group we shall denote by $G_u$ (resp. $G_s$) the set of all unipotent (resp. semisimple) elements of $G$. By $G^0$ we denote the identity component of $G$ and by $G'$ the derived subgroup of $G$. A torus $S$ in $G$ will be called maximal if $S \subset T$ implies that $T = S$ for any torus $T$ of $G$. The center of $G$ is denoted by $Z(G)$. The centralizer of $s \in G$ resp. $S \subset G$ in a subgroup $H \subset G$ will be denoted by $Z_H(s)$ resp. $Z_H(S)$. The existence and uniqueness of the Jordan decomposition for elements of $G$ will be used without explicit reference. All group homomorphisms will be homomorphisms of affine algebraic groups. For other proofs of the structure theorem for connected solvable affine algebraic groups we refer the reader to the references [1-5].

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The group of all invertible $n \times n$ upper triangular matrices will be denoted by $B_n$. Its subgroup consisting of all diagonal matrices is denoted by $D_n$. We have $B_n = U_n \rtimes D_n$ where $U_n$ is the closed connected subgroup of $B_n$ consisting of all unipotent elements of $B_n$.

We start with some preliminary facts.

THEOREM 1 (Lie-Kolchin). *Every connected solvable affine algebraic group can be embedded in some* $B_n$ *as a closed subgroup.* □

COROLLARY. *If* $G$ *is a connected solvable affine group then* $G' \subset G_u$. □

THEOREM 2 (Chevalley). *If* $N$ *is a closed normal subgroup of an affine group* $G$ *then there exists a homomorphism* $f: G \to GL_n(k)$ *such that* $\operatorname{Ker} f = N$. □

For the proofs of Theorems 1 and 2 see, for instance, [5, Theorems 6.7 and 5.1.3].

LEMMA 1. *If* $f: G \to H$ *is a surjective homomorphism of affine algebraic groups and* $N := \operatorname{Ker} f$ *then:*

(i) $f(G^0) = H^0$;

(ii) $f(G_u) = H_u$ *and* $f(G_s) = H_s$;

(iii) $\dim G = \dim N + \dim H$;

(iv) *If* $N$ *and* $H$ *are connected then* $G$ *is connected.*

*Proof.* For the proofs of (i) and (iii) see for instance [4, Section 7.4]. (ii) follows from the fact that $f$ preserves the Jordan decomposition [4, Theorem 2.4.8]. We shall sketch the proof of (iv). Since $N$ is connected, we have $N \subset G^0$. By (i) we have $f(G^0) = H^0 = H$, and consequently $G = NG^0 = G^0$. □

We need a lemma to prove the centralizer theorem. For a more general version of this lemma see [2, Proposition (9.3)].

LEMMA 2. *Let* $N$ *be a closed normal connected abelian unipotent subgroup of an affine group* $G$ *and let* $s \in G_s$. *Then* $M := \{sus^{-1}u^{-1} : u \in N\}$ *is a closed connected subgroup of* $N$, *the multiplication map* $\mu: M \times Z_N(s) \to N$ *is bijective, and* $Z_N(s)$ *is connected.*

*Proof.* Since $N$ is abelian, the map $f: N \to N$, defined by $f(u) = sus^{-1}u^{-1}$, is a morphism of algebraic groups whose kernel is $Z_N(s)$ and image $M$, so $M$ is a closed connected subgroup of $N$. If $x \in M \cap Z_N(s)$ then $x = sus^{-1}u^{-1}$ for some $u \in N$. Since $usu^{-1} = x^{-1}s = sx^{-1}$ is semisimple and $x$ is unipotent, the uniqueness of the Jordan decomposition implies that $x = 1$. Hence $M \cap Z_N(s) = 1$ and so $\mu$ is injective. By Lemma 1 (iii) we have $\dim N = \dim M + \dim Z_N(s)$, which implies that the homomorphism $\mu$ is also surjective, i.e., $MZ_N(s) = N$. The same argument shows that $MZ_N(s)^0 = N$, and so $Z_N(s)$ must be connected. □

THEOREM 3. *If $G$ is a connected solvable affine group and $s \in G_s$ then $Z_G(s)$ is connected and $G = G_u Z_G(s)$.*

*Proof.* We use induction on $\dim G$. If $G$ is abelian the assertions are trivial. Otherwise let $N$ be the last non-trivial term of the derived series of $G$. By the Corollary of Theorem 1, $N$ is unipotent. We now apply Theorem 2 to this $G$ and $N$. Let $f$ be as in that theorem. We shall write $\bar{x}$ for $f(x)$ and $\bar{G}$ for $f(G)$.

Let $z \in G$ be such that $\bar{z} \in Z_{\bar{G}}(\bar{s})$. Then $szs^{-1}z^{-1} \in N$. By Lemma 2 there exists $u \in N$ and $v \in Z_N(s)$ such that $szs^{-1}z^{-1} = sus^{-1}u^{-1} \cdot v$. Since $v$ commutes with $u$ and $s$, and $zsz^{-1} = v^{-1} \cdot usu^{-1}$, it follows that $v = 1$. Thus $u^{-1}z \in Z_G(s)$ and consequently we have a short exact sequence

$$1 \to Z_N(s) \hookrightarrow Z_G(s) \to Z_{\bar{G}}(\bar{s}) \to 1\,.$$

By Lemma 2, $Z_N(s)$ is connected. By Lemma 1 (iii) we have $\dim \bar{G} < \dim G$. By induction hypothesis, we conclude that $Z_{\bar{G}}(\bar{s})$ is connected and that $\bar{G} = (\bar{G})_u \cdot Z_{\bar{G}}(\bar{s})$. Now Lemma 1 (iv) implies that $Z_G(s)$ is connected. By part (ii) of the same lemma we have $f(G_u) = (\bar{G})_u$ and so $f(G_u Z_G(s)) = (\bar{G})_u Z_{\bar{G}}(\bar{s}) = \bar{G}$. Since $N \subset G_u$, it follows that $G = G_u Z_G(s)$. □

We now proceed to prove the main results about the structure of connected solvable affine groups. But first we need two lemmas.

LEMMA 3. *Let $S \subset B_n$ be a commuting set of semisimple elements. Then there exists $b \in B_n$ such that $b^{-1}Sb \subset D_n$.*

*Proof.* It is an elementary fact of linear algebra that there exists $a \in GL_n(k)$ such that $a^{-1}Sa \subset D_n$. Hence if $M_n(k)$ is the algebra of $n$ by $n$ matrices over $k$ and $A$ its subalgebra generated by $S$, we know that $A$ is semisimple (and commutative). Let $V := k^n$ be the space of column vectors and let $e_1, ..., e_n$ be its standard basis. We shall view $V$ as a left $M_n(k)$-module via matrix multiplication. The subspace $V_i$ spanned by the vectors $e_1, ..., e_i$ is an $A$-submodule of $V$ for each $i$. Since $A$ is semisimple, there exist $v_i \in V_i \backslash V_{i-1}$, $1 \leqslant i \leqslant n$, such that $Av_i = kv_i$. Thus if $b$ is the matrix whose $i$-th column is $v_i$, $1 \leqslant i \leqslant n$, then $b \in B_n$ and $b^{-1}Sb \subset D_n$. □

LEMMA 4. *If* $G$ *is a connected solvable affine group,* $T \subset G_s$ *a closed subgroup, and* $G = G_u T$ *then* $T$ *is a torus and* $G = G_u \rtimes T$.

*Proof.* By the Lie-Kolchin theorem we may assume that $G$ is a closed subgroup of some $B_n$. By using the projection map $B_n \to D_n$ we obtain a short exact sequence $1 \to G_u \hookrightarrow G \xrightarrow{p} D \to 1$, where $D \subset D_n$ is a torus. Since $D = p(G) = p(G_u T) = p(T)$, Lemma 1 (i) implies that $p(T^0) = D$. Thus $G = G_u T^0$ and using $T \cap G_u = 1$ we conclude that $T = T^0$. In particular $T$ is abelian and by Lemma 3 we may assume that $T \subset D_n$, i.e., $T = D$. Since $B_n = U_n \rtimes D_n$, $G_u \subset U_n$, $T = D \subset D_n$, and $G = G_u T$, it follows that $G = G_u \rtimes T$. □

THEOREM 4. *Let* $G$ *be a connected solvable affine group. Then* $G = G_u \rtimes T$ *where* $T$ *is a maximal torus. In particular,* $G_u$ *is connected.*

*Proof.* We use induction on dim $G$. Assume first that $G_s \subset Z(G)$. Then $G_s = Z(G)_s$ is a closed subgroup of $G$ and $G = G_u G_s$. The assertion then follows from Lemma 4. Now assume that there exists $s \in G_s \backslash Z(G)$. Then $Z_G(s)$ is a proper closed subgroup of $G$, see e.g. [4, Section 8.2]. By Theorem 3 it is connected and $G = G_u Z_G(s)$. By induction hypothesis there exists a torus $T$ such that $Z_G(s) = Z_G(s)_u T$. Then $G = G_u Z_G(s) = G_u T$ and $G = G_u \rtimes T$ by Lemma 4. □

THEOREM 5. *Let* $G = G_u \rtimes T$ *be a connected solvable affine group. Then every* $s \in G_s$ *is conjugate to an element of* $T$.

*Proof.* We use induction on dim $G$. We have $s = ut$ where $u \in G_u$ and $t \in T$. If $G$ is abelian then $u = 1$ and $s = t$. Otherwise let $N$ be the last non-trivial term of the derived series of $G$. By the corollary of Theorem 1 we have $N \subset G_u$. Hence $N$ is a closed connected normal abelian unipotent subgroup of $G$. By Theorem 2 and the induction hypothesis there exists $x \in G$ such that $xsx^{-1} = tv$ where $v \in N$. By Lemma 2, $v = t^{-1}utu^{-1}z$ where $u \in N$ and $z \in Z_N(t)$. Hence $xsx^{-1} = utu^{-1}z$. Since $xsx^{-1}, utu^{-1} \in G_s$, $z \in G_u$, and $z$ commutes with $u$ and $t$, it follows that $z = 1$ and consequently $xsx^{-1} = utu^{-1}$. □

THEOREM 6. *If* $G = G_u \rtimes T$ *is a connected solvable affine group and* $S \subset G_s$ *is a commuting set then* $Z_G(S)$ *is connected and* $aSa^{-1} \subset T$ *for some* $a \in G$. *In particular, all maximal tori of* $G$ *are conjugate.*

*Proof.* We use induction on $\dim G$. The assertions are obvious if $S \subset Z(G)$. Otherwise choose $s \in S \backslash Z(G)$. By Theorem 5 we may assume that $s \in T$. Then $Z_G(s)$ is a proper closed subgroup of $G$ containing $T$ and $S$. By Theorem 3, $Z_G(s)$ is connected. Since $\dim Z_G(s) < \dim G$, we can apply the induction hypothesis to conclude the proof. □

It is now easy to describe connected nilpotent affine groups.

THEOREM 7. *A connected solvable affine group* $G = G_u \rtimes T$ *is nilpotent if and only if* $G_s = T \subset Z(G)$. *In that case* $G = G_u \times T$.

*Proof.* Assume that $G$ is nilpotent. We prove that $G_s = T \subset Z(G)$ by induction on $\dim G$. We may assume that $G$ is not abelian. Let $N$ be the last non-trivial term of the lower central series of $G$. Let $f$ be as in Theorem 2 and $\bar{G} = f(G)$. Then $\bar{G} = f(G_u T) = (\bar{G})_u f(T)$. By induction hypothesis we have $f(T) = (\bar{G})_s \subset Z(\bar{G})$. Consequently if $t \in T$ and $x \in G$ then $u := txt^{-1}x^{-1} \in N$. Since $N \subset Z(G) \cap G_u$, and $xtx^{-1} = u^{-1}t = tu^{-1}$ we must have $u = 1$. Thus $T \subset Z(G)$ and, by Theorem 5, $G_s = T$. The converse is obvious. □

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Dragomir Ž. Đoković

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

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