

# GLOBAL CONSTRUCTION OF THE NORMALIZATION OF STEIN SPACES

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Objektyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.04.2024**

Persistenter Link: <https://doi.org/10.5169/seals-56603>

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## GLOBAL CONSTRUCTION OF THE NORMALIZATION OF STEIN SPACES

by Sandra HAYES and Geneviève POURCIN

### INTRODUCTION

A fundamental tool in the theory of complex manifolds  $X$  is Riemann's Theorem on Removable Singularities of holomorphic functions which ensures that all functions holomorphic outside of a rare analytic subset of  $X$  and locally bounded on  $X$  can be extended to functions holomorphic on all of  $X$ . In other words, all weakly holomorphic functions on  $X$  are actually holomorphic. Although this theorem does not hold for arbitrary complex spaces, Oka [12] showed in 1951 that every complex space  $X$  can be modified to a complex space  $\tilde{X}$  for which Riemann's continuation theorem is valid, the so-called normalization of  $X$ .

Stein spaces  $X$  are complex spaces which can be completely described by the algebra  $\mathcal{C}(X)$  of global holomorphic functions. Since a complex space is Stein if and only if its normalization is Stein [11], it is natural to ask if the normalization  $\tilde{X}$  of a Stein space  $X$  can be constructed just from the holomorphic functions on  $X$ . Phrased differently, the question is whether the algebra  $\mathcal{C}(\tilde{X})$  of all holomorphic functions on  $\tilde{X}$  or equivalently, the algebra  $\tilde{\mathcal{C}}(X)$  of all weakly holomorphic functions on  $X$ , can be derived from the algebra  $\mathcal{C}(X)$  of holomorphic functions on  $X$ .

The purpose of this paper is to demonstrate that this is possible when  $X$  is irreducible:  $\tilde{\mathcal{C}}(X)$  is the topological closure of the integral closure  $\widetilde{\mathcal{C}(X)}$  of  $\mathcal{C}(X)$ . An example given in § 1 shows that  $\widetilde{\mathcal{C}(X)}$  is not in general topologically closed even if  $X$  is locally irreducible.  $\tilde{\mathcal{C}}(X)$  can also be obtained by taking the intersection of the localizations  $S_x^{-1} \widetilde{\mathcal{C}(X)}$  of the integral closure  $\widetilde{\mathcal{C}(X)}$  of  $\mathcal{C}(X)$  with respect to  $S_x := \{g \in \mathcal{C}(X) : g(x) \neq 0\}$  for every  $x \in X$  (see § 3).

The proof relies on an analytic and an algebraic theorem, namely Rossi's theorem [13] generalizing the Remmert quotient and the integral closure theorem of Mori-Nagata [7].

An analytic consequence of the construction presented here is that the normalization  $\tilde{X}$  of an irreducible Stein space  $X$  is  $\widetilde{\mathcal{O}(X)}$ -convex,  $\widetilde{\mathcal{O}(X)}$ -separable and has local coordinates by functions in  $\widetilde{\mathcal{O}(X)}$ . Some algebraic results are that  $\mathcal{O}(\tilde{X})$  is completely normal and that the two algebras  $\widetilde{\mathcal{O}(X)}$  and  $\mathcal{O}(\tilde{X})$  are always locally equal, i.e. their localizations at all maximal ideals in  $\mathcal{O}(X)$  are equal.

In this paper, a complex space refers to a reduced complex space with countable topology.

### 1. EXAMPLE OF A STEIN SPACE $X$ WITH $\widetilde{\mathcal{O}(X)} \neq \mathcal{O}(\tilde{X})$

Let  $(X, \mathcal{O})$  be a complex space with normalization  $\pi: \tilde{X} \rightarrow X$ . Since  $\pi$  is surjective, the map  $\pi^*: \mathcal{O}(X) \rightarrow \mathcal{O}(\tilde{X})$ ,  $f \mapsto f \circ \pi$ , is injective and the holomorphic functions  $\mathcal{O}(X)$  on  $X$  can be considered to be a subring of the holomorphic functions  $\mathcal{O}(\tilde{X})$  on the normalization  $\tilde{X}$  of  $X$ ; this will be indicated by  $\mathcal{O}(X) \subset \mathcal{O}(\tilde{X})$ . If  $X$  is irreducible and Stein, then  $\mathcal{O}(\tilde{X})$  contains the integral closure  $\widetilde{\mathcal{O}(X)}$  of  $\mathcal{O}(X)$  but does not always coincide with it, as will be shown in this section.

For an irreducible complex space  $(X, \mathcal{O})$ , the integral domain  $\mathcal{O}(X)$  is said to be *normal*, if it is integrally closed in its field of fractions  $Q(\mathcal{O}(X))$ , i.e.  $\widetilde{\mathcal{O}(X)} = \mathcal{O}(X)$ . Recall that  $Q(\mathcal{O}(X))$  is the field of meromorphic functions  $M(X)$  on  $X$  when  $X$  is irreducible and Stein due to Theorem B [10, 53.1, 52.17], and that the algebras  $M(X)$  and  $M(\tilde{X})$  are isomorphic for every complex space  $X$  [8, p. 161].

The following characterization of normal irreducible Stein spaces  $X$  by their global function algebra  $\mathcal{O}(X)$  is essentially contained in [2, § 1, p. 35].

**THEOREM 1.** *An irreducible Stein space  $X$  is normal if and only if the integral domain  $\mathcal{O}(X)$  is normal.*

An analysis of the proof shows that even when  $X$  is just irreducible and normal,  $\mathcal{O}(X)$  is also normal. Theorem 1 implies

**COROLLARY 1.** *For an irreducible Stein space  $X$  with normalization  $\tilde{X}$ , the integral closure  $\widetilde{\mathcal{O}(X)}$  of  $\mathcal{O}(X)$  is contained in  $\mathcal{O}(\tilde{X})$ .*

The following example shows that there are functions  $f \in \mathcal{O}(\tilde{X})$  which are not integral over  $\mathcal{O}(X)$ . In this example,  $X := (\mathbb{C}, \mathcal{O})$  is an irreducible

and locally irreducible Stein space given by a substructure of the canonical complex plane  $(\mathbf{C}, \mathcal{O})$ , which is then the normalization  $\tilde{X}$  of  $X$ . The substructure is defined by a "Strukturausdünnung" (see [10]) which results by replacing the stalks  $\mathcal{O}_n, n \in \mathbf{N}$ , with the stalks of generalized Neil parabolas becoming steeper as  $n$  increases. More precisely, let  $(p_n)_{n \in \mathbf{N}}$  be a strictly increasing sequence of prime numbers. For every  $n \in \mathbf{N}$ ,

$$X_n := \{(x, y) \in \mathbf{C}^2 : x^{p_n} = y^{p_n+1}\}$$

is an irreducible, locally irreducible analytic subset of  $\mathbf{C}^2$  with the origin as the only singularity and with normalization

$$\pi_n: \mathbf{C} \rightarrow X_n, t \mapsto (t^{p_n+1}, t^{p_n}).$$

Let  $f \in \mathcal{O}(\mathbf{C})$  be the identity and denote by  $\mathcal{O}_{X_n}$  the canonical complex structure on  $X_n$ . The germ  $f_0 \in \mathcal{O}_0$  of  $f$  at the origin is integral over  $\mathcal{O}_{X_n,0}$  with respect to a polynomial of degree  $p_n$ , and  $p_n$  is the minimal degree of all such polynomials.

Now define  $X := (\mathbf{C}, \mathcal{O}')$  as a substructure of the canonical plane  $(\mathbf{C}, \mathcal{O})$  with stalks

$$\mathcal{O}'_x \cong \begin{cases} \mathcal{O}_x & , \quad x \notin \mathbf{N} \\ \mathcal{O}_{X_n,0} & , \quad x = n \in \mathbf{N} \end{cases}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}'_n & \rightarrow & \mathcal{O}_n \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{O}_{X_n,0} & \xrightarrow{\pi_n^*} & \mathcal{O}_0, \end{array}$$

where  $\mathcal{O}'_n \rightarrow \mathcal{O}_n$  is the map induced by the identity  $(\mathbf{C}, \mathcal{O}) \rightarrow (\mathbf{C}, \mathcal{O}')$  and  $\mathcal{O}_n \cong \mathcal{O}_0$  is determined by the translation  $\mathbf{C} \rightarrow \mathbf{C}, z \mapsto z - n$ .

The identity  $f \in \mathcal{O}(\mathbf{C})$  is not integral over  $\mathcal{O}'(\mathbf{C})$ , because otherwise every polynomial of integral dependence would have degree at least  $p_n$  for all  $n \in \mathbf{N}$ .

In conclusion it should be mentioned that  $\mathcal{O}(\tilde{X})$  is almost integral over  $\mathcal{O}(X)$  [7, § 3] for every irreducible Stein space  $X$ , since  $X$  has a global universal denominator [10, E.73a].



2. CONSTRUCTION OF  $\mathcal{O}(\tilde{X})$  FROM  $\mathcal{O}(X)$  FOR STEIN SPACES  $X$ 

According to a theorem of Oka [12], the normalization sheaf  $\tilde{\mathcal{O}}$  of weakly holomorphic functions on a complex space  $(X, \mathcal{O})$  is coherent. Consequently, there is a canonical topology making  $\tilde{\mathcal{O}}$  a Fréchet sheaf; the global weakly holomorphic functions  $\tilde{\mathcal{O}}(X)$  will always carry this topology. Since the holomorphic functions  $\mathcal{O}(\tilde{X})$  on the normalization  $\tilde{X}$  of  $X$  are topologically isomorphic to  $\tilde{\mathcal{O}}(X)$  [8, 8.3], the question posed in the introduction can now be answered.

**MAIN THEOREM.** *For an irreducible Stein space  $X$ , the integral closure  $\widetilde{\mathcal{O}(X)}$  of  $\mathcal{O}(X)$  is dense in  $\tilde{\mathcal{O}}(X)$ .*

*Proof.* Let  $\pi: \tilde{X} \rightarrow X$  be the normalization of  $X$  and put  $A := \widetilde{\mathcal{O}(X)}$ . Since  $\pi$  is proper,  $\tilde{X}$  is  $\mathcal{O}(X)$ -convex and therefore  $\bar{A}$ -convex. Note that Corollary 1 implies  $A \subset \tilde{\mathcal{O}}(X)$  and that  $\bar{A}$  is the closure of  $A$  with respect to the canonical topology in  $\tilde{\mathcal{O}}(X)$ .

Consider the equivalence relation  $R$  on  $\tilde{X}$  defined by  $\bar{A}$ , i.e.  $(x, y) \in R$  iff for every  $f \in \bar{A}$ ,  $f(x) = f(y)$ . Rossi's theorem [13] ensures that the topological quotient  $Y := \tilde{X}/R$  can be given the complex structure of a Stein space such that the projection  $p: \tilde{X} \rightarrow Y$  is holomorphic and proper and the map  $p^*: \mathcal{O}(Y) \rightarrow \mathcal{O}(\tilde{X})$ ,  $f \mapsto f \circ p$ , induces an isomorphism  $\mathcal{O}(Y) \cong \bar{A}$ .

It suffices to show that every  $f \in \mathcal{O}(\tilde{X})$  can be factorized through a holomorphic function on  $Y$ , meaning that an  $F \in \mathcal{O}(Y)$  exists with  $F \circ p = f$ . This will be accomplished by first factorizing  $f \in \mathcal{O}(\tilde{X})$  through a continuous function  $F$  on  $Y$  and then proving that  $F$  is actually holomorphic. The existence of such a continuous factor  $F$  for  $f$  is equivalent to demonstrating that every  $f \in \mathcal{O}(\tilde{X})$  is constant on the fibers of  $p$ . The validity of this geometric statement will be shown now using commutative algebra.

$\mathcal{O}(\tilde{X})$  is almost integral over  $\mathcal{O}(X)$  (see § 1), and hence over the localization  $S_x^{-1}A$  of  $A$  with respect to  $S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\}$  for every  $x \in X$ . Moreover,

$$S_x^{-1}A = \widetilde{S_x^{-1}\mathcal{O}(X)}$$

holds [3, V, 1.5, Corollary 1]. The localization  $S_x^{-1}\mathcal{O}(X) = \mathcal{O}(X)_{m(x)}$  of the Stein algebra  $\mathcal{O}(X)$  at the maximal ideal  $m(x) := \{f \in \mathcal{O}(X) : f(x) = 0\}$  is noetherian — even more, it's excellent [2, p. 35]. According to a theorem of Mori-Nagata, the integral closure of a noetherian integral domain is completely normal [7, 4.3, 3.6], implying

$$(*) \quad \mathcal{O}(\tilde{X}) \subset \bigcap_{x \in X} S_x^{-1} A.$$

For  $f \in \mathcal{O}(\tilde{X})$ ,  $a \in \tilde{X}$  and  $b \in p^{-1}(p(a))$ , it is now possible to conclude that  $f(a) = f(b)$  is true. Let  $x := \pi(a)$ . Due to (\*), functions  $g \in S_x$  and  $h \in A$  exist with  $f = h/g \circ \pi$ . Since  $a$  and  $b$  are equivalent with respect to the equivalence relation  $R$ ,  $f(a) = f(b)$  follows, and a continuous function  $F: Y \rightarrow \mathbb{C}$  exists with  $F \circ p = f$ .

Since the Stein complex structure on  $Y$  is not in general the canonical ringed quotient structure, it is still necessary to verify that  $F$  is holomorphic in order to prove the density of  $A$  in  $\mathcal{O}(\tilde{X})$ . To that end, let  $H \in \mathcal{O}(Y)$  and  $G \in \mathcal{O}(Y)$  have the property that  $H \circ p = h$  and  $G \circ p = g \circ \pi$ . Such functions exist because  $p^*(\mathcal{O}(Y)) = \bar{A}$  holds. Then  $F = H/G$  follows, and the germ  $F_{p(a)}$  is the germ of a holomorphic function at  $p(a)$ , since the germ  $G_{p(a)}$  of  $G$  at  $p(a)$  is a unit. The surjectivity of  $p$  implies that  $F$  is holomorphic on  $Y$ , completing the proof of the theorem.

Note that the topology induced by  $\mathcal{O}(\tilde{X})$  on any subalgebra  $A$  of  $\mathcal{O}(\tilde{X})$  is the metrizable topology of uniform convergence on compact subsets of  $X$ . Because the closure  $\bar{A}$  of  $A$  in  $\mathcal{O}(\tilde{X})$  is its completion,  $\bar{A}$  can be obtained without referring directly to  $\mathcal{O}(\tilde{X})$ . Thus the Main Theorem can be stated as follows:

If  $\tilde{X}$  denotes the normalization of an irreducible Stein space  $X$ , then  $\mathcal{O}(\tilde{X})$  is the completion of the integral closure  $\widetilde{\mathcal{O}(X)}$  of  $\mathcal{O}(X)$ .

### 3. APPLICATIONS

In this section  $X$  will denote an irreducible Stein space with normalization  $\pi: \tilde{X} \rightarrow X$ ,  $\widetilde{\mathcal{O}(X)}$  will be the integral closure of the holomorphic functions  $\mathcal{O}(X)$  on  $X$ ,  $\mathcal{O}(\tilde{X})$  the Fréchet algebra of weakly holomorphic functions on  $X$  (or equivalently, the Fréchet algebra of holomorphic functions  $\mathcal{O}(\tilde{X})$  on  $\tilde{X}$ ), and

$$S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\} \quad \text{for } x \in X.$$

Although the example given in the first section shows that the algebras  $\widetilde{\mathcal{O}(X)}$  and  $\mathcal{O}(\tilde{X})$  are not always equal, the inclusion (\*) in the proof of the Main Theorem implies that they are locally equal in the following sense.

**THEOREM 2.** For every  $x \in X$ , the localizations of  $\widetilde{\mathcal{O}(X)}$  and  $\mathcal{O}(\tilde{X})$  with respect to  $S_x$  coincide.

The next theorem implies an algebraic description of the topological closure of  $\widetilde{\mathcal{O}(X)}$  in  $\tilde{\mathcal{O}}(X)$ .

THEOREM 3.  $\mathcal{O}(\tilde{X}) = \bigcap_{x \in X} S_x^{-1} \widetilde{\mathcal{O}(X)}.$

*Proof.* Let  $f \in M(\tilde{X}) = M(X)$  be such that for every  $x \in X$  there is a  $g \in S_x$  and an  $h \in \widetilde{\mathcal{O}(X)}$ , with  $f = h/g \circ \pi$ . Then the germ  $f_a$  of  $f$  at an arbitrary point  $a \in \tilde{X}$  is holomorphic, because the germ of  $g \circ \pi$  at  $a$  is a unit. Hence  $f \in \mathcal{O}(\tilde{X})$ , and the assertion is proved.

COROLLARY 2. The topological closure of  $\widetilde{\mathcal{O}(X)}$  in  $\tilde{\mathcal{O}}(X)$  is the intersection of the localizations of  $\widetilde{\mathcal{O}(X)}$  with respect to  $S_x$  for all  $x \in X$ .

The next result characterizes the weakly holomorphic functions on  $X$  as being exactly those meromorphic functions on  $X$  which are almost integral over  $\mathcal{O}(X)$ .

COROLLARY 3.  $\mathcal{O}(\tilde{X})$  is completely normal.

*Proof.* Let  $f \in M(\tilde{X})$  be almost integral over  $\mathcal{O}(\tilde{X})$ . Then  $f$  is almost integral over  $\mathcal{O}(X)$  and therefore over  $S_x^{-1} \widetilde{\mathcal{O}(X)}$  for every  $x \in X$  which has been shown to be completely normal in the proof of the Main Theorem. An application of Theorem 3 yields  $f \in \mathcal{O}(\tilde{X})$  and hence the assertion.

Using the classical Oka-Weil-Cartan Theorem [1, Anhang zu VI, § 4], an immediate consequence of the Main Theorem is

THEOREM 4.  $\tilde{X}$  is  $\widetilde{\mathcal{O}(X)}$ -convex,  $\widetilde{\mathcal{O}(X)}$ -separable and has local coordinates by functions in  $\widetilde{\mathcal{O}(X)}$ .

A property which ensures that the holomorphic functions on  $\tilde{X}$  are integral over the holomorphic functions on  $X$  is that  $\mathcal{O}(\tilde{X})$  is a finite  $\mathcal{O}(X)$ -module.

THEOREM 5. Let  $u \in \mathcal{O}(X)$  be any global universal denominator for  $X$ . Then  $\mathcal{O}(\tilde{X})$  is isomorphic to the closed ideal  $u\mathcal{O}(\tilde{X})$  in  $\mathcal{O}(X)$ , and  $\mathcal{O}(\tilde{X})$  is a finite  $\mathcal{O}(X)$ -module if and only if this ideal is finitely generated.

*Proof.* Recall that a global universal denominator  $u$  for  $X$  always exists [10, E.73a]. The multiplication map

$$\mathcal{O}(\tilde{X}) \rightarrow \mathcal{O}(X), \quad f \mapsto uf,$$

defines an injective  $\mathcal{O}(X)$ -module homomorphism. Thus,  $\mathcal{O}(\tilde{X})$  is isomorphic to the ideal  $u\mathcal{O}(\tilde{X})$  in  $\mathcal{O}(X)$  which will now be denoted by  $I$ . Consider the transporter ideal  $J := \tilde{\mathcal{O}} : \frac{1}{u} \mathcal{O}$  of  $\frac{1}{u} \mathcal{O}$  into  $\tilde{\mathcal{O}}$  which is a coherent sheaf of ideals in  $\tilde{\mathcal{O}}$ . The global sections  $J(X)$  form a closed ideal of  $\mathcal{O}(X)$  by a theorem of Cartan [4, 5], due again to the fact that  $X$  is Stein. Because  $J(X) = I$  holds, the assertion follows.

COROLLARY 4. *If  $\mathcal{O}(\tilde{X})$  does not coincide with  $\widetilde{\mathcal{O}(X)}$ , the closed ideal  $u\mathcal{O}(\tilde{X})$  in  $\mathcal{O}(X)$  is not finitely generated.*

In a Stein algebra  $\mathcal{O}(X)$ , every finitely generated ideal is closed, as Cartan [4, 5] showed. If  $X$  is at least two-dimensional, Forster [6] gave examples of closed ideals in  $\mathcal{O}(X)$  which are not finitely generated. According to Corollary 4, the space constructed in § 1 gives a one-dimensional example.

## REFERENCES

- [1] BEHNKE, H. und P. THULLEN. *Theorie der Funktionen mehrerer komplexen Veränderlichen*, 2. edition. Springer-Verlag, Berlin, Heidelberg, New York, 1970.
- [2] BINGENER, J. und U. STORCH. Resträume zu analytischen Mengen in Steinschen Räumen. *Math. Ann.* 210 (1974), 33-53.
- [3] BOURBAKI, N. *Algèbre commutative*. Hermann, Paris, 1969.
- [4] CARTAN, H. *Séminaire*. E.N.S. 1951/1952.
- [5] ———. Idéaux et modules de fonctions analytiques de variables complexes. *Bull. Soc. Math. France* 78 (1950), 28-64.
- [6] FORSTER, O. Zur Theorie der Steinschen algebren und Moduln. *Math. Zeitschr.* 97 (1967), 376-405.
- [7] FOSSUM, R. *The divisor class group of a Krull domain*. Ergebnisse der Math. und ihrer Grenzgebiete 74, Springer-Verlag, Berlin, Heidelberg, New York, 1973.
- [8] GRAUERT, H. and R. REMMERT. *Coherent analytic sheaves*. Grundle. Math. Wiss. 265. Springer-Verlag, Berlin, Heidelberg, New York, 1984.
- [9] ———. *Theorie der Steinschen Räume*. Grundle. Math. Wiss. 227. Springer-Verlag, Berlin, Heidelberg, New York, 1977.

- [10] KAUP, L. and B. KAUP. *Holomorphic functions of several variables*. De Gruyter, New York, 1983.
- [11] NARASIMHAN, R. A note on Stein spaces and their normalizations. *Ann. Scuola Norm. Sup. Pisa* 16 (1962), 327-333.
- [12] OKA, K. Sur les fonctions analytiques de plusieurs variables, VIII, Lemme fondamental. *J. Math. Soc. Japan* 3 (1951), 204-214, 259-278.
- [13] ROSSI, H. Analytic spaces with compact subvarieties. *Math. Ann.* 146 (1962), 129-145.

(Reçu le 13 mars 1988)

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