§6. Examples

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$\tilde{K}_i \cap \tilde{B}_{\mathbf{R}}$ is a subgroup of lower dimension, we have $\tilde{K}_i = \overline{\tilde{K}_i \cap \tilde{B}_{\mathbf{R}}\, s_i \tilde{B}_{\mathbf{R}}}$. The Iwasawa decomposition of $\tilde{G}_{\mathbf{R},i}$ shows that $\tilde{K}_i \tilde{B}_{\mathbf{R}} = \tilde{G}_{\mathbf{R},i} \tilde{B}_{\mathbf{R}}$. Now $\mathcal{O}_i = \mathbf{B}_{\mathbf{R}} \quad \mathbf{B}_{\mathbf{R}}\, s_i \mathbf{B}_{\mathbf{R}}$, and $B_{\mathbf{R}}\, s_i \mathbf{B}_{\mathbf{R}} = U_{\mathbf{R},i} s_i B_{\mathbf{R}}$, where $U_{\mathbf{R},i}$ corresponds to the positive roots β_i and (if $2\beta_i$ is a root) $2\beta_i$. Since $U_{\mathbf{R},i} \leqslant \tilde{G}_{\mathbf{R},i}$, this completes the proof of (2.12). Note $\mathcal{O}_i/\tilde{B}_{\mathbf{R}} = \tilde{K}_i/\tilde{K}_i \cap \tilde{B}_{\mathbf{R}}$. Since $U_{\mathbf{R},i}$ is homeomorphic to a real vector space of dimension $n_i = m_{\beta i} + m_{2\beta i}$, and $\mathcal{O}_i/\tilde{B}_{\mathbf{R}}$ is compact, we also conclude that $\mathcal{O}_i/\tilde{B}_{\mathbf{R}}$ is a sphere of dimension n_i, and that $\mathcal{O}_i \to \mathcal{O}_i/\tilde{B}_{\mathbf{R}}$ has a local section. This completes the proof of Theorem 5.3. □

Now let $\mathscr{B}_{G/K}$ be the building associated to the topological Tits system of (5.3). To prove Theorem 5.1, it is enough to show (as in § 4):

(5.4) THEOREM (Quillen). *$(\Omega_{alg}G)^\tau$ acts freely on $\mathscr{B}_{G/K}$, with orbit space G/K.*

Proof. $B_{G/K}$ is a quotient space of $(\Omega_{alg}G)^\tau \times K/C_K t_m \times \Delta$, where Δ is the Cartan simplex in $\mathfrak{t}_{\mathfrak{m}}$ (here we are using (5.2); note that $(L_{alg}G)^\tau \cap P^\tau = G^\sigma = K$). Hence the orbit space of the $(\Omega_{alg}G)^\tau$-action is a quotient of $K/C_K t_m \times \Delta$. As in the proof of (4.2), we see that the equivalence relation here coincides with that of Theorem 1.9. Hence the orbit space is G/K, as desired. To see that the action is free, we introduce the space of special paths $\mathscr{S}_{G/K}$ path of the form $f(e^{2\pi i t}) \exp tX$ with $f \in (\Omega_{alg}G)^\tau$ and $X \in \mathfrak{m}$. The proof now proceeds exactly as in (4.2); details are left to the reader. □

The other results of § 4 also go through: $\mathscr{S}_{G/K}$ is $(L_{alg}G)^\tau$—equivariantly homeomorphic to the building $\mathscr{B}_{G/K}$, and if $X, Y \in m$, $\exp X = \exp Y$ implies $\exp tX = f \exp tY$, where $f \in (\Omega_{alg}G)^\tau$.

§ 6. EXAMPLES

In this section we discuss six examples, the first four of which arise in the Bott periodicity theorems (§ 7). The first and last examples are discussed in some detail, the others are only sketched.

(6.1) $\Omega(SU(2n)/Sp(n))$. This is perhaps the simplest nonsplit example. $SU(2n)$ has an involution σ given by $\sigma(A) = J\bar{A}J^{-1}$, where J is the matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. The fixed group K is $Sp(n)$. The extension of σ to $SL(2n, \mathbf{C})$

is given by the same formula, so the corresponding real form is $SL(n, \mathbf{H}) \equiv GL(n, \mathbf{H}) \cap SL(2n, \mathbf{C})$. For convenience we now make the obvious change of basis transforming J into a direct sum of 2×2 matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

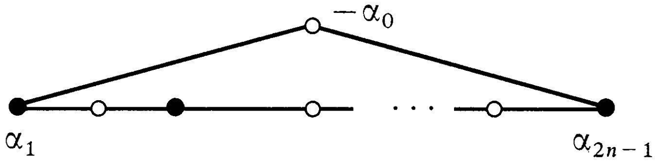
In this basis $\mathfrak{t}_{\mathfrak{m}}$ consists of the diagonal matrices $a = \begin{pmatrix} a_1 & & & & \\ & a_1 & & & \\ & & \ddots & & \\ & & & a_n & \\ & & & & a_n \end{pmatrix}$

with the a_i pure imaginary.

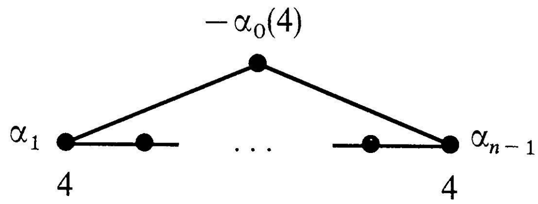
Hence $C_K\mathfrak{t}_{\mathfrak{m}} = \prod_1^n Sp(1)$, $N_K\mathfrak{t}_{\mathfrak{m}} = \sum_n \int Sp(1)$, and the relative Weyl group $W_{G,K}$ is $\sum_n$. The root systems are described as follows. In the usual notation, the root system Φ of $SU(2n)$ consists of

$$\{\pm (e_i - e_j) : 1 \leqslant i, j \leqslant 2n, i \neq j\}.$$

Clearly $\Phi_0 = \{\pm (e_i - e_{i+1}) : i \text{ odd}\}$. If $a \in \mathfrak{t}_{\mathfrak{m}}$ is as above, let $f_i(a) = a_i$. Then the restricted root system Σ consists of $\{\pm (f_i - f_j) : 1 \leqslant i, j \leqslant n : i \neq j\}$, and so has type A_{n-1}. Moreover it is clear that the multiplicities are all equal to four. Thus the extended Satake diagram is



and the extended Dynkin diagram is



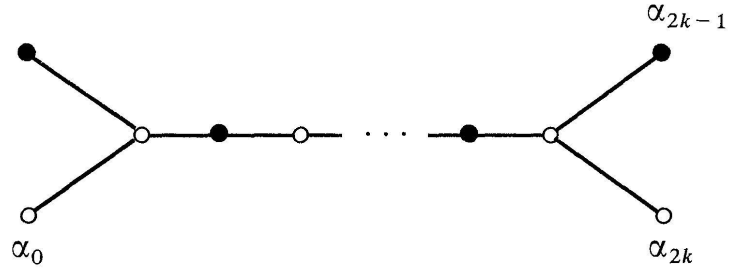
Note that the parabolic subgroup Q (obtained from the black nodes of the Satake diagram) is just the isotropy group of the standard flag $\mathbf{C}^2 \subset \mathbf{C}^4 \cdots \subset \mathbf{C}^{2n-2} \subset \mathbf{C}^{2n}$. The corresponding "quasi-Borel" subgroup Q^σ (minimal parabolic, in the standard terminology) is then the isotropy group of the complete quaternionic flag $\mathbf{H}^1 < \mathbf{H}^2 \cdots < \mathbf{H}^n$ (in $SL(n, \mathbf{H})$). The little K_α subgroups ($\alpha \in \Sigma$) are all $Sp(2)$'s.

Now consider the involution τ on $L_{alg}SL(2n, \mathbf{C}) = SL(2n, \mathbf{C}[z, z^{-1}])$. If $f(z) = \sum A_k z^k$, $(\tau f)(z) = \sum J\bar{A}_k J^{-1} z^k$. Hence the fixed group L^{τ}_{alg} is just $SL(n, \mathbf{H}[z, z^{-1}])$. Since we know that the affine Weyl group $\tilde{W}$ of type A_{n-1} has $P_{\tilde{W}/W}(t) = \prod_{i=1}^{n} (1-t^i)^{-1}$, the extended Dynkin diagram above shows immediately that $\Omega SU(2n)/Sp(n))$ has torsion—free homology, with Poincaré series $\prod_{i=1}^{n} (1-t^{4i})^{-1}$. For more applications of this approach, see [9] and § 7.

(6.2) $\Omega(SO(2n)/U(n))$. For convenience we take $n = 2k$, $k \geqslant 2$. Let J be as in (6.1) and define $\sigma(A) = JAJ^{-1}(A \in SO(2n))$. Then $K \cong U(n)$, embedded as the matrices $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. Now make the same change of basis as in (6.1). Then $\mathfrak{t}_{\mathfrak{m}}$ consists of matrices

$$A = \begin{pmatrix} A_1 & & & & & \\ & -A_1 & & & & \\ & & \ddots & & & \\ & & & & A_k & \\ & & & & & -A_k \end{pmatrix}$$

where $A_i = \begin{pmatrix} 0 & -a_i \\ a_i & 0 \end{pmatrix}$. Since the original root system Φ consists of $\{\pm e_i \pm e_j : 1 \leqslant i, j \leqslant n, i \neq j\}$, where e_i denotes projection on the ith 2×2 block in $\mathfrak{t}$, clearly $\sum$ has type C_k and consists of $\pm (f_i, -f_j)$, $\pm 2f_i$, where $f_i(A) = a_i$. We have $\Phi = \{\pm (e_i + e_{i+1}) : i \text{ odd}\}$ and $W_{G,K} = \sum_n \int \sum_2$. The simple roots $f_i - f_{i+1}$ have multiplicity 4, whereas $2f_i$ has multiplicity one. Thus the extended Satake diagram is



and the extended Dynkin diagram is

(Here the usual basis $e_1, -e_2, e_2 - e_3, \cdots e_{n-1} - e_n, e_{n-1} + e_n$ for Φ has been replaced by the basis

$$e_1 + e_2, -c_2 - e_3, e_3 + e_4, -e_4 - e_5, \cdots e_{n-1} + e_n, e_{n-1} - e_n .$$

In particular the highest root is now $e_1 - e_2$).

(6.3) $(\Omega(SU(n)/SO(n))$. Here the involution on $SU(n)$ is $\sigma(A) = \bar{A}$. Hence we are in the split case and everything is transparent:

$$G_{\mathbf{R}} = SL(n, \mathbf{R}), (L_{alg}\, SL(n, \mathbf{C}))^{\tau} = SL(n, \mathbf{R}[z, z^{-1}]), \text{ etc.}$$

The Satake and Dynkin diagrams are just the Dynkin diagram for A_{n-1} (all Satake nodes white, all multiplicities equal one). For further details and applications, see [9].

(6.4) $(\Omega(Sp(n)/U(n))$. Embed $Sp(n)$ in $SU(2n)$ as usual and define $\sigma(A) = \bar{A}$. The fixed group is $U(n)$ embedded as matrices $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ with A, B real. Again we see that we are in the split case; the associated real form $G_{\mathbf{R}}$ is $Sp(n, \mathbf{R})$, L^{τ}_{alg} is $Sp(n, \mathbf{R}[z, z^{-1}]$, etc. The extended Dynkin diagram is

$$\begin{array}{ccccccc} \alpha_0 & & & & & & \alpha_n \\ \bullet & \Rightarrow & \bullet & \cdots & \bullet & \Leftarrow & \bullet \\ 1 & & 1 & & 1 & & 1 \end{array}$$

We can conclude e.g. that $\Omega Sp(n)/U(n)$ has mod 2 Poincaré series $\prod_{i=1}^{n} (1-t^{2i-1})^{-1}$ (cf. Theorem 5.9).

(6.5) ΩS^n. Assume $n = 2k+1$; the case n even is similar. Define an involution σ on $SO(2k+1)$ by $\sigma(A) = \varepsilon A \varepsilon^{-1}$, where

$$\varepsilon = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

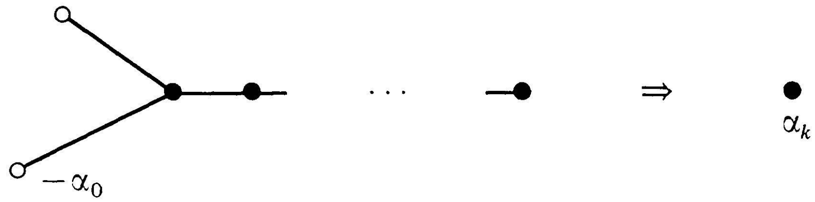
Then $K = S(O(1) \times O(2k)) \cong O(2K)$, so $K' = SO(2k)$. The corresponding real from $G'_{\mathbf{R}}$ consists of matrices $\begin{pmatrix} a_{11} & \cdots \\ \vdots & A \end{pmatrix}$ in $SO(2k+1, \mathbf{C})$ with a_{11} and A real and the remaining entries pure imaginary. In fact (as in easily checked) $G'_{\mathbf{R}} \cong SO(1, 2k)$. The torus $\mathfrak{t}_{\mathfrak{m}}$ is the set of matrices

$$\begin{pmatrix} 0 & -a & \\ a & 0 & \\ & & 0 \end{pmatrix}$$

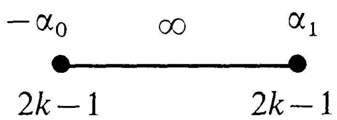
and hence the relative Weyl group has order 2

$$\left(\text{generated by} \begin{pmatrix} -1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}\right).$$

Using the usual notation for Φ, $\Phi_0 = \{\pm (e_i - e_j), \pm e_k : i, j, k \neq 1\}$. Thus Σ has type A_1 (no doubled roots) and the multiplicity of its one positive root is $2k - 1$. The extended Satake diagram is



and the extended Dynkin diagram is



(The symbol ∞ indicates that $s_0 s_1$ has infinite order.) The groups $\tilde{K}_0, \tilde{K}$, are both $SO(2k)$'s. In particular we obtain a model for ΩS^n with one cell in each dimension of the form $i(n-1)$.

(6.6) $\Omega \mathbf{C}P^{n-1}$. This example serves to illustrate two phenomena not encountered above: a nontrivial involution on the Satake diagram, and a restricted root system which is not reduced. Take $G = SU(n)$ and define $\sigma(A) = \varepsilon A \varepsilon$, where ε is as in (6.5). Thus $K = S(U(1) \times U(n-1))$ and $G/K = \mathbf{C}P^{n-1}$. The corresponding real form of $SL(n, \mathbf{C})$ is denoted $SU(1, n-1)$ and is described as in (6.5): matrices $\begin{pmatrix} a_{11} & \cdots \\ \vdots & A \end{pmatrix}$ in $SL(n, \mathbf{C})$ with a_{11}, A real and the remaining entries pure imaginary. The torus $\mathfrak{t}_{\mathfrak{m}}$ consists of matrices

$$\begin{pmatrix} 0 & a & \\ a & 0 & \\ & & 0 \end{pmatrix}$$

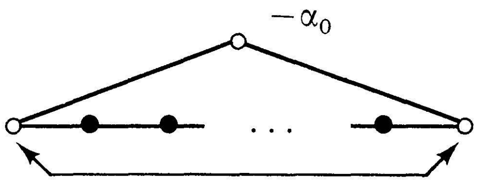
with a pure imaginary. Here we are taking as Cartan subalgebra in $\mathfrak{su}(n)$ the matrices

$$\begin{pmatrix} a & b & & & \\ b & a & & & \\ & & c_3 & & \\ & & & \ddots & \\ & & & & c_n \end{pmatrix}.$$

Using this Cartan subalgebra, a simple system of roots $\alpha_1, \cdots \alpha_{n-1}$ for Φ is given by the following table:

α_1	$2a + b - c_3$	
α_2	$-2a + c_3 - c_4$	
α_i	$c_{i+1} - c_{i+2}$	$(3 \leqslant i \leqslant n-2)$
α_{n-1}	$b + c_n$	

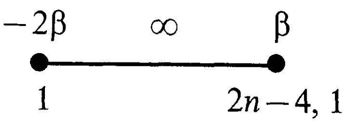
The highest root $\alpha_0 = \alpha_1 + \alpha_2 + \dots + \alpha_{n-1}$ then takes the value $2b$. The action of σ on these roots is given by $\alpha_i \mapsto -\alpha_i (2 \leqslant i \leqslant n-2)$ and $\sigma\alpha_1 = \alpha_2 + \alpha_3 + \cdots + \alpha_{n-1}$. Thus Φ_0 is the span of $\alpha_2, \cdots \alpha_{n-2}$, and the extended Satake diagram is



Furthermore the restricted root system Σ has type BC_1 (type A_1 with doubled root). Indeed if β is defined by

$$\beta\begin{pmatrix} 0 & b & \\ b & 0 & \\ & & 0 \end{pmatrix} = b,$$

we see that β has multiplicity $2n-4$ and 2β has multiplicity one (α restricts to 2β). Hence the extended Dynkin diagram is



Following the procedure discussed in § 5, we have at once that G_β is all of $SU(n)$, so $K_\beta = K \cong U(n-1)$. Note $K/C_K\mathfrak{t}_{\mathfrak{m}} = S^{2n-3}$. On the other hand $K_{2\beta} \cong SO(2)$ ($G_{2\beta}$ is the $SU(2)$ in the upper left corner). From the Dynkin diagram we conclude that our model for $\Omega \mathbf{C}P^{n-1}$ has one cell in each of the dimensions 0, 1, $2n-2$, $2n-1$, $4n-4$, $4n-3$, ... in other words, the cell series is $(1+t)(1+t^{2n-2})^{-1}$. (Recall that the affine Weyl group of type $\tilde{A}_1$ is just the free product $\mathbf{Z}/2 * \mathbf{Z}/2$, so that the Bruhat cells are indexed by $1, s_0, s_1 s_0, s_0 s_1 s_0$, etc. By the above remarks, s_0 receives weight one and s_1 weight $2n-3$, hence our formula.)

§ 7. Bott Periodicity

Bott's theorem, in its original form [6], is a general statement about the range in which certain maps $K/L \xrightarrow{\varphi} \Omega G/K$ are homotopy equivalences. The periodicity theorems proper are then deduced from this, taking G, K, L to be suitable classical groups. In this section we derive a version of Bott's theorem by showing that in many cases the map φ is a homeomorphism onto a Schubert subspace of $\Omega(G/K)$; then one merely counts cells. In fact, in these cases we will be able to read off the desired range directly from the Dynkin diagram of G/K.

We assume that G is simple and simply-connected. (As usual, the essential point is that G/K is simply-connected; then we can if necessary replace G by its universal cover.) Let $\lambda: [0, 1] \to G$ be a path of the form $\lambda(t) = \exp tX$, where X belongs to the coweight lattice J_m. In otherwords, $X \in t_{\mathfrak{m}}$ and $\exp X$ is central in G. Then for all $k \in K$, the path $\varphi_\lambda \equiv \lambda k \lambda^{-1} k^{-1}$ actually lies in $(\Omega_{alg} G)^\tau$; see the proof of 4.2. Hence $\lambda \mapsto \varphi_\lambda$ defines a *Bott map* $K/C_k\lambda \xrightarrow{\varphi} (\Omega_{alg} G)^\tau$ ($\cong \Omega G/K$). Identifying J_m with the group of paths λ as above, the most interesting λ are obviously the fundamental coweights ε_i dual to the simple restricted roots β_i: $\beta_j(\varepsilon_i) = \delta_{ij} (1 \leqslant i, j \leqslant l)$. Among these one may single out the very convenient class of *miniscule coweights*. These are the ε_i dual to a *miniscule root* β_i-i.e. a simple root which occurs with coefficient one in the highest root β_0. The miniscule coweights are precisely the nonzero elements of the coweight lattice which are also vertices of the Cartan simplex. They exist whenever the root system is reduced and not of type G_2, F_4 or E_8; in terms of the Dynkin diagram, they correspond to nodes on the ordinary diagram which are conjugate to the special node $-\alpha_0$ under an automorphism of the extended diagram. Thus for example in type A_n every simple root is miniscule,