§2. Torsions of chain complexes and manifolds

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The non-trivial case of Theorem 2 is the case $l_1 = l_2 = \ldots = l_{n-1} = 0$: otherwise $u(L) \geqslant u$ so that $\Delta_{u-1}(L) = 0$ and we may put $\lambda = 0$.

The proof of Theorems 1, 2 goes along the same lines as the proof of the formula (2) given in [4]. These proofs are based on a relationship between the Alexander polynomials and Reidemeister-type torsions, established in [4]. This relationship is recalled in § 2. In § 3 several easy algebraic lemmas are proved. Theorems 1, 2 are proved in § 4.

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§ 2. Torsions of chain complexes and manifolds

2.1. The torsion of a chain complex (see [3]). Let Q be a field. If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are two bases of a Q-module then $a_i = \sum_{j=1}^{n} c_{i,j} b_j$ where $(c_{i,j})$ is a non-singular $n \times n$-matrix over Q; the determinant $\det(c_{i,j}) \in Q \backslash 0$ is denoted by $[a/b]$.

Let $C = (C_m \to \cdots \to C_0)$ be a chain Q-complex. Suppose that each Q-module C_i is finite dimensional with a preferred basis c_i and each Q-module $H_i(C)$ also has a preferred basis h_i. (The case $C_i = 0$ or $H_i(C) = 0$ is not excluded; by definition the zero module has the empty basis.) In this setting one defines the torsion $\tau(C) \in Q$ as follows. For each $i = 1, 2, \ldots, m$ choose a sequence $b_i = (b^i_1, \ldots, b^i_{r_i})$ of elements of C_i such that $\partial_{i-1}(b_i) = (\partial_{i-1}(b^i_1), \ldots, \partial_{i-1}(b^i_{r_i}))$ is a basis in $\operatorname{Im}(\partial_{i-1}: C_i \to C_{i-1})$. For each $i = 0, 1, \ldots, m$ choose a lifting $\tilde{h}_i$ of the basis h_i to $\operatorname{Ker} \partial_{i-1}$. The combined sequence $\partial_i(b_{i+1})\tilde{h}_i b_i$ is a basis in C_i. (It is understood that $b_0 = \emptyset$ and $b_{m+1} = \emptyset$). Put

$$(3) \qquad \tau(C) = \prod_{i=0}^{m} [\partial_i(b_{i+1})\tilde{h}_i b_i / c_i]^{\varepsilon(i)}$$

where $\varepsilon(i) = (-1)^{i+1}$. Clearly, $\tau(C) \in Q \backslash 0$. It is easy to verify that $\tau(C)$ does not depend on the choice of b_i and $\tilde{h}_i$.

(Note that the torsion of C defined in Milnor's survey article [3] equals $\pm\, \tau(C)^{-1} \in Q/\pm 1$ and that Milnor uses the additive notation for the multiplication in $Q \backslash 0 = K_1(Q)$.)

2.1.1. Lemma (multiplicativity of torsion). *Let* $0 \to C' \to C \to C'' \to 0$ *be a short exact sequence of m-dimensional chain complexes over a field* Q.

Suppose that for all $i = 0, 1, \ldots, m$ *the modules* C_i, C'_i, C''_i *are provided with preferred bases* c'_i, c_i, c''_i *which are compatible, in the sense that* $[c'_i c''_i / c_i] = \pm 1$. *Suppose that for all* $i = 0, 1, \ldots, m$ *the homology modules* $H_i(C), H_i(C'), H_i(C'')$ *are provided with preferred bases. Let* $\mathscr{H}$ *be the homology sequence of the sequence* $0 \to C' \to C \to C'' \to 0$:

$$\mathscr{H} = \big(H_m(C') \to H_m(C) \to \cdots \to H_0(C) \to H_0(C'')\big) .$$

Consider $\mathscr{H}$ *as an acyclic based chain complex over* Q. *Then* $\tau(C) = \pm\, \tau(C')\tau(C'')\tau(\mathscr{H})$.

For a proof see [3].

2.2. THE TORSION ω. Let M be an orientable compact smooth manifold of odd dimension m with $\operatorname{rg} H_1(M) \geqslant 1$. Denote the free abelian group $H_1(M)/\operatorname{Tors} H_1(M)$ by G. Denote the fraction field of the group ring $\mathbf{Z}[G]$ by Q. Provide Q with the involution $q \mapsto \bar{q}$ which sends $g \in G$ to g^{-1}. The field Q defines via the natural homomorphism $\mathbf{Z}[\pi_1(M)] \to Q$ a system of local coefficients on M. We shall denote this system by the same symbol Q. Assume that $H_*(\partial M ; Q) = 0$. In this setting one can consider a torsion-type invariant $\omega(M)$ of M which is "an element of $Q \backslash 0$ defined up to multiplication by $\pm\, g q \bar{q}$ with $g \in G$ and $q \in Q \backslash 0$" (see [4]).

Recall the definition of $\omega(M)$ given in [4, § 5]. Let $\tilde{M} \to M$ be the regular covering of M corresponding to the kernel of the natural homomorphism $\pi_1(M) \to G$. Fix a C^1-triangulation of M and the induced G-equivariant triangulation of $\tilde{M}$. Choose over each simplex of the (fixed) triangulation of M a simplex of the triangulation of $\tilde{M}$. These simplices in $\tilde{M}$ being arbitrarily oriented and ordered determine "natural" bases of the modules of the simplicial chain $\mathbf{Z}[G]$-complex $C_*(\tilde{M} ; \mathbf{Z})$. These bases induce "natural" Q-bases in the chain Q-complex

$$C = Q \otimes_{\mathbf{Z}[G]} C_*(\tilde{M} ; \mathbf{Z}) .$$

For all $i = 0, 1, \ldots, m$ choose an arbitrary Q-basis h_i in $H_i(M ; Q) = H_i(C)$. Denote by $\tau(C, h_0, \ldots, h_m)$ the torsion of C with respect to the bases in chain modules constructed above and the bases $h_0, h_1, \ldots, h_m$ in homology. Since $H_*(\partial M ; Q) = 0$ the semi-linear intersection form $H_i(M ; Q) \times H_{m-i}(M ; Q) \to Q$ is non-singular. Let v_i be the matrix of this form regarding the bases h_i and h_{m-i}. Put

$$d = \tau(C, h_0, h_1, \ldots, h_m) \prod_{i=0}^{r} (\det v_i)^{-\varepsilon(i)} \in Q \backslash 0$$

where $r = (m-1)/2$ and $\varepsilon(i) = (-1)^{i+1}$. It is easy to show that under a different choice of natural bases and bases $h_0, h_1, ..., h_m$ the element d is replaced by $\pm g q \bar{q} d$ with $g \in G$, $q \in Q \backslash 0$. Thus the set $\{\pm g q \bar{q} d \mid g \in Q \backslash 0\} \subset Q$ does not depend on the choice of bases. It also does not depend on the choice of triangulation in M. It is this set which is $\omega(M)$.

An explicit formula established in [4] enables us to calculate $\omega(M)$ in terms of the orders of $\mathbf{Z}[G]$-modules $H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; \mathbf{Z})$, $H_*(\tilde{M}) = H_*(\tilde{M}; \mathbf{Z})$ and related modules. (The notion of the order of a module is recalled in Sec. 3.1.) Denote by J the image of the inclusion homomorphism $H_r(\partial\tilde{M}) \to H_r(\tilde{M})$ where $r = (m-1)/2$. Then up to multiples of type $q\bar{q}$ with $q \in Q \backslash 0$

$$\omega(M) = \operatorname{ord}\left(\operatorname{Tors}_{\mathbf{Z}[G]} H_r(M, \partial M)\right) (\operatorname{ord} J)^{\varepsilon(r)} \prod_{i=0}^{r-1} [\operatorname{ord} H_i(\partial M)]^{\varepsilon(i)} \tag{4}$$

(see [4, Theorem 5.1.1]). Note that the equalities $Q \otimes_{\mathbf{Z}[G]} H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; Q) = 0$ imply that $H_*(\partial\tilde{M})$ and J are torsion $\mathbf{Z}[G]$-modules. Therefore $\operatorname{ord} H_i(\partial\tilde{M})$ and $\operatorname{ord} J$ are non-zero elements of $\mathbf{Z}[G]$.

We shall apply formula (4) in the case where M is the exterior of an n-component link $K \subset S^m$ with odd m. The condition $H_*(\partial M; Q) = 0$ is always fulfilled in this case. Here the field Q is canonically identified with the field of rational functions of n variables $Q_n = Q(t_1, ..., t_n)$. Thus $\omega(M) \subset Q_n$. If $m \geqslant 5$ then (4) implies that

$$\Delta(K)(t_1, ..., t_n) \cdot \prod_{i=1}^{n} (t_i - 1) \subset \omega(M).$$

If $m = 3$ then there exists a unique subset $\alpha = \alpha(K)$ of the set $\{1, 2, ..., n\}$ such that

$$\Delta_{u(K)}(K)(t_1, ..., t_n) \cdot \prod_{i \in \alpha} (t_i - 1) \subset \omega(M).$$

For proofs and details consult [4, § 5].

§ 3. Algebraic lemmas

3.1. Preliminary definitions. For a finitely generated module H over a (commutative) domain R we denote by $\operatorname{rk}_R H$ or, briefly, by $\operatorname{rk} H$ the integer $\dim_Q(Q \otimes_R H)$ where $Q = Q(R)$ denotes the field of fractions of R. For a R-linear homomorphism $f: H \to H'$ we put $\operatorname{rk} f = \operatorname{rk}_R f(H)$. Note that if $\bar{R}$ is the localization of R at some multiplicative system then $Q(\bar{R}) = Q(R)$ and therefore the (exact) functor $(H \mapsto \bar{R} \otimes_R H, f \mapsto \operatorname{id}_{\bar{R}} \otimes f)$