§2. Topological Buildings

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

$y = s_1 \cdots s_k (s_i \in S)$ and x has a reduced decomposition obtained by deleting some subset of the s_i's occuring in y. (For a very nice account of these related matters, see [14]). If W is finite, W has a unique element w_0 of maximal length, we define the length of W to be $l(w_0)$.

§ 2. Topological Buildings

A *Tits system* (G, B, N, S) consists of a group G, subgroups B and N, and a set S, which satisfy the following axioms:

(2.1) $B \cap N$ is normal in N, and S is a set of involutions generating $W \equiv N/B \cap N$,

(2.2) B and N generate G,

(2.3) If $s \in S$, $sBs \neq B$,

(2.4) if $s \in S$, $w \in W$, then $sBw \leqslant BwB \cup BswB$.

(The use of expressions such as sBw is a standard abuse of notation).

Example. Let G be a reductive algebraic group over an algebraically closed field (e.g., $GL(n, \mathbf{C})$), let B be a Borel subgroup (e.g. upper triangular matrices), and let N be the normalizer of a maximal torus (that lies in B). This data determines a set S of simple reflections generating the Weyl group W (e.g., the usual generators $s_1, \ldots, s_{n-1}$ of Σ_n). Then one of the main results in the structure theory of reductive groups is that (G, B, N, S) is a Tits system (see for example [15]).

Throughout this paper we will assume that the set S is finite; its cardinality l is the *rank* of the system.

We next list some of the important properties of a Tits system.

(2.5) (Bruhat Decomposition) $G = \coprod_{w \in W} BwB$ (disjoint union),

(2.6) (W, S) is a Coxeter system.

A subgroup P of G is *parabolic* if it contains a conjugate of B. In particular if $I \subseteq S$, the subgroup P_I generated by B and I is parabolic.

(2.7) (a) The parabolic subgroups containing B are precisely the P_I, $I \subseteq S$. No two of these are conjugate; in particular there are exactly 2^l such subgroups, which form a lattice isomorphic to the lattice of subsets of S.

(b) $P_I = BW_IB$

(c) Every parabolic P is self-normalizing: $N_G P = P$.

(2.8) (Bruhat decomposition, general version) $G = \coprod_{w \in W_I \backslash W / W_J} P_I w P_J$ (disjoint union).

The next result, which we will refer to as the *Steinberg Lemma*, is somewhat technical; however it is not hard to prove and is extremely useful. It is a mild generalization of Theorem 15 of [32] and Proposition 3.1 of [19].

(2.9) Let $I \subseteq S$ and suppose w is the unique element of minimal length of wW_I. Suppose $w = w_1 \ldots w_k$ where $l(w) = l(w_1) + \ldots + l(w_k)$. Then

(a) If Y_i is any subset of Bw_iB such that $Y_i \to Bw_iB/B$ is bijective (resp. surjective) $(1 \leqslant i \leqslant k)$, then $Y_1 \times Y_2 \ldots \times Y_k \to BwP_I/P_I$ is bijective (resp. surjective).

(b) Suppose $w_i \in S$, $1 \leqslant i \leqslant k$ i.e., $w_1 \ldots w_k$ is a reduced decomposition of w). Let Z_i, $1 \leqslant i \leqslant k$, be any subset containing 1 of P_{w_i} such that $Z_i \to P_{w_i}/B$ is surjective. Then the image of $Z_1 \times \cdots Z_k \to G/P_I$ is $\coprod_{x \leqslant w} BxP_I/P_I$.

The maps in (a), (b) are the obvious multiplication/projection maps. Part b refers to the Bruhat order on W^I.

(2.10) *Remark.* The Tits system of a reductive algebraic group has several additional features: $B = HU$, where H is a maximal torus and U is a normal unipotent subgroup, U in turn is described in terms of its root subgroups, and there is an "opposite" Borel subgroup B^- such that $B \cap B^- = H$. This additional structure can also be axiomatized in an elegant way, leading to the "refined" Tits system of Kac and Peterson [19]. One then obtains, for example, the *Birkhoff decomposition* $G = \coprod_{w \in W} B^- wB$ as a consequence of the axioms.

We now define a *topological* Tits system to be a Tits system such that G is a topological group, B and N are closed subgroups, and W is discrete (i.e. $N \cap B$ is an open subgroup of N). We will usually also assume (for reasons which will be apparent shortly):

(2.11) *Axiom.* If I is a proper subset of S, W_I is finite.

This axiom is satisfied if W is an irreducible affine Weyl group, or finite. To get any interesting results some further axiom seems necessary. One direction is considered in [11], where the groups in question are algebraic groups over local fields, with the valuation topology. Here, with loop groups in mind, the following axiom seems efficient:

(2.12) *Axiom.* For each $s \in S$ there is a subset A_s of P_s such that (a) $A_s B = P_s$, (b) A_s is compact and contains 1, and (c) $A_s = \overline{A_s \cap BsB}$. This axiom is motivated by Steinberg's approach [32].

(2.13) PROPOSITION. *Let (G, B, N, S) be a topological Tits system satisfying (2.12). Then*

(a) $\overline{BwB} = \coprod_{x \leqslant w} BxB (w \in W)$. *More generally if* $I \leqslant S$, *and* $w \in W^I$, $\overline{BwP_I} = \coprod_{x \leqslant w} BxP_I$ *(here* $x \in W^I$*)*,

(b) *B-orbits in* G/P_I *are locally closed,*

(c) *If* W *satisfies (2.11), parabolic subgroups are closed.*

Proof. First we show $P_s = \overline{BsB}$. Since $P_s = A_s B$, with A_s compact and B closed, P_s is closed, so $P_s \geqslant \overline{BsB}$. But also $B \subset P_s = A_s B \subset \overline{BsB}$, which proves our claim. Part (a) now follows easily from the Steinberg lemma: Let $M_w = \coprod_{x \leqslant w} BxP_I$, and let $w = s_1 \cdots s_k$ be a reduced decomposition. Then $M_w = A_1 \cdots A_k P_I$ and hence is closed. Next, suppose $x \leqslant w$; we must show $BxB \leqslant \overline{BwB}$. It is enough to consider the case when X has a reduced decomposition $x = s_1 \cdots \hat{s}_i \cdots s_k$ (omit s_i). Then

$$BxP_I = A'_1 \cdots A'_{i-1} A'_{i+1} \cdots A'_k P_I \leqslant A'_1 \cdots A'_{i-1} \bar{A}_i \cdots A'_k P_I \leqslant \overline{BwP_I}$$

(since $1 \in A_i$), where $A'_i = A_i \cap Bs_iB$. This proves (a). Part (b) is immediate since the complement of BwP_I in its closure is a finite union of sets of the form M_x, hence is closed. Since $P_I = BW_IB$, (c) is also immediate from (a) if W_I is finite. □

From now on we will assume 2.11 and 2.12. The homogeneous spaces G/P_I will be called *flag spaces.* The B-orbits $E_w = BwP_I/P_I$ are *Schubert strata* and the compact subspaces $\overline{E_W}$ are *Schubert subspaces.*

We next consider the *building* $\mathscr{B}_G$ associated to a topological Tits system (G, B, N, S). (The notation is ambiguous—indeed in the case of loop groups, G will support two natural but totally different Tits system. However the system we have in mind will be clear from the context.) In the discrete case, $\mathscr{B}_G$ is usually defined as the following simplicial complex. The vertices are the maximal (proper) parabolics, and $P_1 \cdots P_k$ span a simplex if $\bigcap_{i=1}^{k} P_i$ contains a conjugate of B. In general it is convenient to reinterpret this definition as follows: first of all, by definition every parabolic P is conjugate to a unique P_I; we say that P has type I. Thus the maximal parabolics are the parabolics of type $[s]$, where $[s] = S - \{s\}$. More generally the k-simplices correspond to the parabolics of type I, where $|I| = l - k - 1$. Thus the simplices all have dimension $\leqslant l - 1$, with the $l - 1$ simplices corresponding to the conjugates of B. Furthermore, in view of 2.7 (c), the set of parabolics of type I is canonically identified with G/P_I — xP_I corresponding to xP_Ix^{-1}. One can casily check that with this interpretation, a simplex xP_I is a face of a simplex yP_J if and only if $I \supset J$ and $xP_I = yP_I$. In particular, every simplex is a face of some $l-1$ simplex. Hence, as a set, B_G can be identified with $G/B \times \Delta/\sim$, where Δ is the $l-1$ simplex with vertex set S, and $(g_1B, X_1) \sim (g_2B, X_2)$ if $X_1 = X = X_2$, $X \in \Delta_I$, and $g_1P_I = g_2P_I$. (Here Δ_I is the face of Δ corresponding to $I \leqslant S$.) We will therefore *define* the building $\mathscr{B}_G$ associated to the topological Tits system (G, B, N, S) to be $G/B \times \Delta$ modulo this equivalence relation, with the quotient topology.

Remark. Another way of expressing this is as follows: Let C be the category defined by the poset of proper subsets of S (including the empty set). We have a functor from C to topological spaces given by $I \mapsto G/P_I$. Then $\mathscr{B}_G$ is precisely the homotopy colimit of this diagram of spaces, in the sense of [8], p. 327 ff.

(2.14) PROPOSITION. *The equivalence relation on* $G/B \times \Delta^{l-1}$ *is generated by the relations* $(g_1B, X) \sim (g_2B, X)$ *if* X *lies on the wall* Δ_s *and* $g_1P_s = g_2P_s$.

Proof. In the usual language, (2.14) is the assertion that any two chambers are linked by a "gallery". (See e.g. [11], appendix.) Since the action of G on G/B induces a well defined action on $\mathscr{B}_G$, we are reduced to showing that if $(B, X) \sim (gB, X)$—i.e. $X \in \Delta_I$ and $g \in P_I$—then (B, X) and (gB, X) are linked by a sequence of relations of the stated type. But $gB = bwB$ with $w \in W_I$; hence if $w = s_1 \cdots s_k$ is a reduced decomposition, the elements $(B, X), (bs_1B, X), (bs_1s_2B, X), \ldots (bwB, X)$ provide the desired sequence. □

Note that the set Δ is a fundamental domain for the action of G on $\mathscr{B}_G$. On the other hand, it is easy to check that the closed subspace $\mathscr{B}_W$ consisting of the pairs $(wB, X), w \in W$, is a fundamental domain for the B action. (The point is that if $bw_1P_I = w_2P_I$, then $w_1P_I = w_2P_I$, by the Bruhat decomposition.) This space $\mathscr{B}_W$, which we will call the *foundation* of the building, is a simplicial complex since W is discrete. Since it will turn out that $\mathscr{B}_G$ is in a sense a "thickening" of the foundation, the following well known description of $\mathscr{B}_W$ may be of interest.

(2.15) PROPOSITION. *Suppose* Φ *is an irreducible root system in the Euclidean space* V. *Then*

(a) *If* W *is the affine Weyl group associated to* Φ, *then* $\mathscr{B}_W$ *is isomorphic as a simplicial* W*-complex to* V *(triangulated by the hyperplanes of* Φ*).*

(b) *If* W *is the Weyl group of* Φ, $\mathscr{B}_W$ *is isomorphic as simplicial* W*-complex to the unit sphere of* V, *triangulated by the Weyl chambers. More precisely,* $\mathscr{B}_W$ *can be identified with the* W *orbit of the outer wall of the Cartan simplex.*

Proof. For (a), map $W \times \Delta \xrightarrow{\varphi} V$ by identifying Δ with the Cartan simplex in V and using the action map. Then φ is onto (1.1) and furthermore $\varphi(w_1, x) = \varphi(w_2, X_2)$ if and only if $X_1 = X = X_2, X \in \Delta_I$, and $w_1 = w_2$ modulo the isotropy group of X. But this isotropy group is precisely W_I (1.2), so φ factors through the desired isomorphism $\mathscr{B}_W \to V$. The proof of (b) is similar. □

We now come to the main result of this section. Filter G/B by $F_k(G/B) = \coprod_{l(w) \leqslant k} E_w$. Similarly, $\mathscr{B}_G$ is filtered by $F_k(\mathscr{B}_G) = F_k(G/B) \times \Delta/\sim$.

(2.16) THEOREM. *Let* (G, B, N, S) *be a topological Tits system which either is discrete or satisfies (2.11) and (2.12). Assume also that the inclusions* $F_k(B_G) \subset F_{k+1}(B_G)$ *are cofibrations. Then*

(a) *If* W *is infinite,* $\mathscr{B}_G$ *is contractible.*

(b) *If* W *is finite of length* r, $\mathscr{B}_G$ *is homotopy equivalent to the* $(l-1)$ *st suspension* $S^{l-1} \wedge (F_r(G/B))/F_{r-1}(G/B))$.

Remark. If G is discrete, $F_k\mathscr{B}_G$ is a subcomplex of the simplicial complex $\mathscr{B}_G$, so the cofibration hypothesis is automatically satisfied. Furthermore if W is finite the smash product in (b) is just a wedge of $| F_rG/B - F_{r-1}G/B |$ $(l-1)$-spheres. This case is due to Solomon and Tits; cf. [11].

Proof of (2.16). Let X_k denote $F_k\mathscr{B}_G/F_{k-1}\mathscr{B}_G$, and let $X'_k = F_k(G/B)/F_{k-1}(G/B)$. Then we will show

(2.17) If k is less than the length of W, X_k is contractible. If $k = r$ = length of W, X_k is homeomorphic to $(F_r(G/B)/F_{r-1}(G/B) \wedge S^{l-1})$.

If W is infinite, it follows that $F_k\mathscr{B}_G$ is contractible for all k, and hence $\mathscr{B}_G$ is contractible. If W is finite, part (b) of the theorem is also immediate.

To prove 2.17, first consider the quotient map $\pi: F_k(G/B) \times \Delta \to X_k$. In fact π is merely collapsing a subspace to a point:

(2.18) Let $A_1 = (b_1w_1B, X_1)$, $A_2 = (b_2w_2B, X_2)$. If $\pi(A_1) = \pi(A_2)$, then either $A_1 = A_2$ or $\pi(A_1) = \pi(A_2) = *$ ($*$ is the basepoint $F_{k-1}B_G$).

For suppose $\pi(A_1) \neq *$, and $X_1 = \mathring{\Delta}_I$. Then $l(w_1) = k$ and $w_1 \in W^I$. This forces $X_1 = X_2$ and $w_1 = w_2 \bmod W_I$; hence $w_1 = w_2$ since $l(w_2) \leqslant k$ by assumption. Then $b_1w_1P_I = b_2w_1P_I$. But whenever $w \in W^I, b_1wP_I = b_2wP_I$ implies $b_1wB = b_2wB$ (easy exercise).

It now follows that $X_k = \bigvee_{l(w)=k} X_w$, where X_w is the image of $\bar{E}_w \times \Delta$ in X_k, and to prove (2.17) we need only consider a fixed X_w. Let $X'_w = \bar{E}_w/(\bar{E}_w - E_w)$, and let Δ' be the subcomplex of Δ consisting of the walls Δ_s such that $l(ws) < l(w)$. Then (2.18) implies:

(2.19) $X_w = X'_w \wedge (\Delta/\Delta')$.

For X_w is $\bar{E}_w \times \Delta$ modulo the subspace of points which are equivalent (in $\mathscr{B}_G$) to a point of lower filtration, namely, $\bar{E}_w \times \Delta' \cup \bar{E}_w - E_w \times \Delta$. It remains to identify Δ'. Since $F_0\mathscr{B}_G = \Delta$ is contractible, we may assume $k \geqslant 1$; then Δ' is nonempty. If $k < l(W)$, then there is at least one $s \in S$ such that $l(ws) > l(w)$; hence Δ' is not the entire boundary of Δ and Δ/Δ' is contractible. If $k = l(W)$, then w is unique, $\Delta' =$ boundary of Δ, and $\Delta/\Delta' = S^{l-1}$. This completes the proof of (2.17), and of the theorem. □

Remark. Our proof of Theorem 2.16 is an adaptation of the standard (discrete) proof to the topological setting. Much of the proof depends only on the Weyl group W, and indeed shows e.g. for W infinite that the foundation of the building is contractible. In fact the deformation of $F_k(\mathscr{B}_W)$ into $F_{k-1}(\mathscr{B}_W)$ has the property that the isotropy group in B of a point X in $\mathscr{B}_W$ is an increasing function of time, and hence extends uniquely to a B-equivariant deformation of $F_k(B_G)$. In the discrete case this extension is automatically continuous, and shows that Theorem (2.16) holds B-equivariantly. (This was observed, (not for the first time) in [21], and has an interesting application concerning the Steinberg representation of a finite Chevalley group.) However this proof does not work in the topological case; simple counterexamples show that the extension will be discontinuous.

In many cases the Bruhat decomposition of G/P is in fact a CW decomposition. The following axioms are convenient in this regard:

(2.20) *Axiom.* For each $s \in S$, the projection $P_s \to P_s/B$ has a local section.

(2.21) *Axiom.* For each $s \in S$, P_s/B is homeomorphic to a sphere of positive dimension.

We then have:

(2.22) THEOREM. *Let* (G, B, N, S) *be a topological Tits system satisfying axioms 2.11, 2.20 and 2.21. Let* $P \equiv P_I$ *be a parabolic subgroup,* $I \leqslant S$, *and give* G/P *the compactly generated topology. Then*

(a) *Axiom 2.12 is satisfied.*

(b) *The Bruhat decomposition of* G/P *is a* CW *decomposition, and the closure relations on the cells are given by the Bruhat order on* W^I.

(c) *The building* $\mathscr{B}_G$ *satisfies the cofibration condition of Theorem 2.16.*

Proof. By assumption there are maps $D^{m(s)} \overset{\varphi_s}{\to} P_s/B$ such that $\varphi_s^{-1}(B) = \partial D^{m(s)}$ and $D^{m(s)}/\partial D^{m(s)} \to P_s/B$ is a homeomorphism. Furthermore φ_s lifts to a map $\tilde{\varphi}_s \colon D^{m(s)} \to P_s$ with $1 \in \tilde{\varphi}_s(\partial D^{m(s)})$. Thus, in Axiom (2.12) we may take $A_s = \tilde{\phi}_s(\mathring{D}^{m(s)})$, proving (a). Since P is closed (2.13c), G/P is a Hausdorff space. If $w \in W^I$ has reduced decomposition $w = s_1 \cdots s_k$, the Steinberg lemma (2.9) shows that the multiplication map $D^{m(s_1)} \times \cdots \times D^{m(s_k)} \to \bar{E}_w$ (using $\tilde{\varphi}_{s_i}$) is a characteristic map for the cell E_w. The boundary of each cell is a finite union of cells of lower dimension by 2.13a, and G/P has the weak topology by assumption. The closure relations also follow from (2.13). This proves (b). For (c) we observe that $\mathscr{B}_G$ (with the compactly generated topology) is itself a CW-complex, and the filtrations $F_k\mathscr{B}_G$ are subcomplexes: Indeed if we regard $\mathscr{B}_G$ as a quotient space of $\coprod_{I \leqslant S}(G/P_I \times \Delta_I)$, it is clear that there is one cell for each $I < S$ and $w \in W^I$. □

If G, P_I are as in the above theorem, and $w \in W^I$ has reduced decomposition $w = s_1 \cdots s_k$, let $n(w) = n(s_1) + \cdots + n(s_k)$. Thus $n(w) = \dim E_w$ and so in particular is independent of the choice of reduced decomposition. Now whenever a space has a locally finite cell decomposition, we have a *cell series* $\sum a_i t^i$, where a_i is the number of cells of dimension i. We then have:

(2.23) COROLLARY. G/P_I *admits a* CW—*decomposition with cell series* $\sum_{w \in W^I} t^{n(w)}$. □

Note also:

(2.24) COROLLARY. *If* W *is finite with maximal length element* w_0, $\mathscr{B}_G$ *is a sphere of dimension* $n(w_0) + l - 1$. □

We conclude this section with two "classical" examples. Let G be a semisimple compact Lie group and consider the Tits system (G, B, N, S), where B is a Borel subgroup, etc. First we claim that this is a topological Tits system satisfying all four of our axioms. Since W is finite, (2.11) is trivially satisfied. In (2.12) we can take A_s to be the "little $SU(2)$" (or $PSU(2)$) G_s (P_s has Iwasawa decomposition $P_s = G_s B$). In any case there is a commutative diagram

$$\begin{array}{ccc} G_s & \to & P_s \\ \downarrow & & \downarrow \\ \mathbf{C}P^1 = G_s/G_s \cap T & = & P_s/B \end{array}$$

which proves (2.20), (2.21), and hence (2.12) simultaneously. The Bruhat decomposition of $G_{\mathbf{C}}/P_I$, P_I parabolic, is then the classical Schubert cell decomposition of the flag variety $G_{\mathbf{C}}/P_I$. We have $n(s) = 2$ for all s, so $n(w) = 2l(w)$ for all $w \in W^I$. In particular the associated building $\mathscr{B}_{G_{\mathbf{C}}}$ is a sphere of dimension $2l(w_0) + l - 1$ (since $l(w)_0$ is the number of positive roots, this is exactly dim $G - 1$).

The second example (which is a generalization of the first) involves symmetric spaces G/K and the associated semisimple real Lie group $G_{\mathbf{R}}$ as in § 1. Thus $G_{\mathbf{R}}$ is the fixed group of the involution σ on $G_{\mathbf{C}}$. Now σ need not preserve the Borel subgroup B of $G_{\mathbf{C}}$, but it does preserve the parabolic Q associated to the black nodes of the Satake diagram. We will write $B_{\mathbf{R}}$, $N_{\mathbf{R}}$, $W_{\mathbf{R}}$, $S_{\mathbf{R}}$ for Q^σ, $N_K t_m$, $W_{G/K}$, $S_{G/K}$, respectively.

(2.25) THEOREM. $(G_{\mathbf{R}}, B_{\mathbf{R}}, N_{\mathbf{R}}, S_{\mathbf{R}})$ *is a topological Tits system satisfying the four axioms.* □

A proof that this is a Tits system can be found in [33]. The parabolic subgroups of $G_{\mathbf{R}}$ are related in an obvious way to those of $G_{\mathbf{C}}$: Given $I \subset S_{\mathbf{R}}$, let I' be the corresponding set in S (see § 1). We denote by $\mathcal{O}_I$ the parabolic in $G_{\mathbf{R}}$ generated by $B_{\mathbf{R}}$ and I. Then $\mathcal{O}_I = (P_{I'})^\sigma$. ($B_{\mathbf{R}}$ is usually called a "minimal parabolic", but this terminology conflicts with our use of the term. From the point of view of Tits systems, it is precisely analogous to the Borel subgroup of $G_{\mathbf{C}}$—although in general it is neither solvable nor connected.) The rest of the theorem is also easily deduced from [33]; the details will be omitted, but see § 5. The main point is that for the minimal parabolics $\mathcal{O}_i$, $\mathcal{O}_i/B_{\mathbf{R}}$ is a sphere of dimension n_i.

As for the building, one can deduce from (2.24) that it is a sphere whose dimension is dim $G/K - 1$. However it is an interesting fact, that does not seem to appear in the literature, that the building can be canonically identified with the "tangent cut locus" of G/K: first recall (cf. [10], [20]) that if M is a compact Riemannian manifold and p is a fixed point of M, a point

x is a *cut point* (with respect to p) if there is a geodesic from p to x that minimizes arc length up to x but no further. The *cut locus* is the set of cut points. Similarly a vector X in the tangent space T_p is a tangent *cut point* if $\exp_p X$ is a cut point along the geodesic $\exp_p(tX)$. The *tangent cut locus* is the set of all such points in T_p, and is homeomorphic to the unit sphere in T_p. When $M = G/K$ we take $p = 1$.

(2.26) THEOREM. *Let G/K be a simply-connected symmetric space, with G simple. Then the tangent cut locus is precisely the K-orbit in $\mathfrak{m}$ of the outer wall of the Cartan simplex Δ_m. It is therefore canonically identified with the topological building of the associated real form $G_{\mathbf{R}}$.*

As usual, the assumption G simple is just for convenience. We sketch the proof: the first assertion is a fairly easy consequence of Theorem (1.8), and is left to the reader. Now consider the building $\mathscr{B}_{G_{\mathbf{R}}}$. It is a quotient space of $G_{\mathbf{R}}/B_{\mathbf{R}} \times \Delta_0 = K/C_K t_m \times \Delta_0$, where Δ_0 is a simplex of dimension (rank G/K)-1; we take Δ_0 to be the outer wall of $\Delta_{\mathfrak{m}}$. For each $I \leqslant S_{G/K}$, let Δ_I temporarily denote the corresponding face of Δ_0; *i.e.* $\{X \in \Delta_0 : \alpha_i(x) = 0 \ \forall\, i \in I\}$. Then the K-orbit of Δ_0 in $\mathfrak{m}$, $K\Delta_0$, is also a quotient of $K/C_K t_{\mathfrak{m}} \times \Delta_0$. The relations are $(k_1 X) \sim (k_2 X)$ if $X \in \mathring{\Delta}_I$ and $k_1 = k_2 \bmod K_I$. But $K_I = K \cap \mathcal{O}_I$, so these relations are identical to the ones that define the building. □

§ 3. LOOP GROUPS

Let LG, $LG_{\mathbf{C}}$ denote the free loop spaces. Let $G_{\mathbf{C}}$ denote the group of loops which are restrictions of regular maps $\mathbf{C}^* \to G_{\mathbf{C}}$, and let $L_{alg}G = L_{alg}G_{\mathbf{C}} \cap LG$. Thus if we fix an embedding $G_{\mathbf{C}} \subset GL(n, \mathbf{C})$, $L_{alg}G$ consists of the loops f in LG admitting a finite Laurent expansion $f(z) = \sum_{k=-m}^{m} A_k z^k$, whereas $L_{alg}G_{\mathbf{C}}$ consists of the loops f in $LG_{\mathbf{C}}$ such that both f and f^{-1} admit finite Laurent expansions. We will also write $\tilde{G}_{\mathbf{C}}$ for $L_{alg}G_{\mathbf{C}}$. In fact $\tilde{G}_{\mathbf{C}}$ is the group of points over $\mathbf{C}[z, z^{-1}]$ of the algebraic group $G_{\mathbf{C}}$. Its Lie algebra is the loop algebra $\tilde{g}_{\mathbf{C}}$ of regular maps $\mathbf{C}^* \to g_{\mathbf{C}}$. The integer m in the above Laurent expansion defines a filtration of $\tilde{G}_{\mathbf{C}}$ by finite dimensional subspaces; we give $\tilde{G}_{\mathbf{C}}$ the corresponding weak topology.

Let P denote the subgroup of $\tilde{G}_{\mathbf{C}}$ consisting of regular maps $\mathbf{C} \to G_{\mathbf{C}}$ (i.e. maps with nonnegative Laurent expansion, or $G_{\mathbf{C}[z]}$), and let $\tilde{B}$ denote the Iwahori subgroup: $\{f \in P : f(0) \in B^-\}$. Finally, let $\tilde{N} = L_{alg}N_{\mathbf{C}}$, and recall that $\tilde{W}$ can be regarded as a "subgroup" of $\tilde{G}_{\mathbf{C}}$, since $R \leqslant \mathrm{Hom}(S^1, T) \leqslant L_{alg}T$. More precisely, we have $\tilde{N}/T_{\mathbf{C}} = \hat{W}$, and $\tilde{W} \subset \hat{W}$.