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ABOUT THE PROOFS OF CALABI'S CONJECTURES ON COMPACT KÄHLER MANIFOLDS

by Ph. DELANOË and A. HIRSCHOWITZ

ABSTRACT

The main part in the proof of Calabi's conjectures consists in *a priori* estimates of order zero, two, three. We explain how a reduction to these estimates may be performed in the framework of C^∞ functions and how higher order estimates may be derived without Schauder's elliptic theory. The main tool is an "elliptic" inverse function theorem [22] [11].

0. INTRODUCTION

T. Aubin [1, 2, 3] and S. T. Yau [23, 24] have brought positive answers to the so-called Calabi's conjectures [6], namely,

THEOREM 0.1. (Aubin, Yau). *On a compact (connected) Kähler manifold with negative first Chern class, there exists a unique Kähler-Einstein metric* g' *satisfying:* $\mathrm{Ricci}(g') = -g'$.

THEOREM 0.2. (Yau). *On any compact (connected) Kähler manifold, given a cohomology class* $c \in H^2(X, \mathbf{R})$ *which contains a Kähler form, every 2-form in the first Chern class is the Ricci form of some Kähler form of* c.

Mathematicians from several fields are concerned with these results, whose main consequences are listed in [23] and in [5] sections 2 and 3. Unfortunately, the proofs are quite technical, they involve rather "irregular" mathematical objects such as elliptic equations with non smooth coefficients, and they make a decisive use of Schauder theory. The aim of the present

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note is to analyze how far these tools are necessary for the proof. It turns out that it is possible to reduce the contribution of elliptic theory mainly to a suitable local inverse function theorem for nonlinear elliptic operators acting on smooth functions [22] [11].

The proof presented below deals only with the reduction to the crucial estimates of order zero, two and three, already obtained by Aubin and Yau. Although it is not so clear in [21] [24] these estimates were performed essentially through coordinate free tensor calculus. We show how higher order estimates may be obtained in the same way.

The whole approach applies as well to the corresponding *real* elliptic Monge-Ampère equation on compact Riemannian manifolds [9] and to various generalizations of it. We shall freely use arguments of Calabi [6, 7, 8], Aubin [1, 2, 3], Yau [23, 24], Bourguignon *et al.* [5] [21].

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1. THE MONGE-AMPÈRE EQUATION

Let X be a compact connected finite-dimensional Kähler manifold. ω denotes the original C^∞ Kähler form, g the corresponding Kähler metric, $\varphi \in C^\infty(X)$ denotes a C^∞ real-valued function on X, and we set

$$\omega' = \omega + \sqrt{-1}\,\partial\bar{\partial}\varphi$$

where ∂ and $\bar{\partial}$ are the usual first order differential operators. Let g' denotes the Kähler metric corresponding to ω'. In the sequel, "smooth" means C^∞.

If g and g' are viewed as morphisms from the antiholomorphic tangent bundle into the holomorphic cotangent bundle T^*, then $(g'g^{-1})$ is an endomorphism of T^* the determinant of which, $\det(g'g^{-1})$ is a smooth function on X. The function φ is said to be *admissible* if and only if $\det(g'g^{-1})$ is strictly positive on X. One proves easily that if φ is admissible, then g' is again a (positive definite Kähler) metric e.g. [2], p. 119.

Let $\lambda \in [0, +\infty)$. It is convenient to denote by A_λ the subset of $C^\infty(X)$ consisting in all admissible real-valued smooth functions φ on X, satisfying, in case $\lambda = 0$ the further zero average condition

$$\int_X \varphi\, dX_g = 0,$$

where dX_g denotes the volume form associated with g. When $\lambda > 0$, A_λ is an *open* subset of $C^\infty(X)$: indeed, the natural injection $C^\infty(X) \hookrightarrow C^2(X)$ is continuous, with respect to the Fréchet and Banach topologies; A_λ is the pull back, by this injection of the *open* set

$$\{\varphi \in C^2(X)\,,\ \det(g'g^{-1}) > 0\}\,.$$

Definition 1.1. Let X be a smooth compact manifold, V a smooth vector bundle on X, $C^\infty(X, V)$ the Fréchet space of smooth sections of V. A *LCFC submanifold* of $C^\infty(X, V)$, is a locally closed finite codimensional Fréchet submanifold of $C^\infty(X, V)$.

The set A_0 is an *open* subset of the LCFC submanifold

$$\{\varphi \in C^\infty(X)\,,\ \int_X \varphi dX_g = 0\}\,.$$

We define the map P_λ, from A_λ to $C^\infty(X)$, by

$$P_\lambda(\varphi) = \lambda\varphi - \operatorname{Log}\det(g'g^{-1})\,.$$

The proofs of theorems 0.1 and 0.2 have been reduced to the solution, when $\lambda \geqslant 0$, of the following complex Monge-Ampère equation (e.g. [21], (lecture n° V), [4] p. 143):

$$(1) \qquad P_\lambda(\varphi) = f\,,$$

where $f \in C^\infty(X)$ is *given*, and in case λ vanishes, has to satisfy the natural constraint (e.g. [1] p. 403, [24] p. 361, [21] p. 85),

$$\int_X e^{-f} dX_g = \int_X dX_g\,.$$

In any case, f ranges in a connected LCFC submanifold B_λ of $C^\infty(X)$. To see that B_0 is *connected*, notice that $0 \in B_0$ and that given any $f \in B_0$, the following path connects f to 0 in B_0:

$$t \in [0, 1] \to f_t =: tf + \operatorname{Log}(\int_X e^{-tf} dX_g) - \operatorname{Log}(\int_X dX_g)\,.$$

The derivative of the map P_λ at $\varphi \in A_\lambda$, is given by

$$(2) \qquad dP_\lambda(\varphi, \delta\varphi) = (\Delta' + \lambda)\,\delta\varphi$$

where Δ' stands for the Laplace operator on functions in the metric g' [21] p. 96. Classically, it follows from the Maximum Principle [20], the Fredholm Alternative theorem and from the elliptic regularity theory, that $dP_\lambda(\varphi, \cdot)$ is *invertible* $\forall \varphi \in A_\lambda$, either from $C^\infty(X)$ to itself when $\lambda > 0$, or from $\{u \in C^\infty(X), \int u dX_g = 0\}$ to $\{v \in C^\infty(X), \int v dX_{g'} = 0\}$ ($dX_{g'}$ denotes the volume form in the metric g') when $\lambda = 0$.

For completeness, let us indicate how, for instance theorem 0.2, can be reduced to equation (1) with $\lambda = 0$. It is quite straightforward. First of all we are given a cohomology class $c \in H^2(X, \mathbf{R})$ such that there exists a Kähler form ω in c; let ρ be the Ricci form of ω: $\rho \in C_1(X)$, the first Chern class of X.

Then we are *given* $\gamma \in C_1(X)$ and hence $f \in C^\infty(X)$ a real function (defined up to an additive constant), which measures the deviation for ω from satisfying 0.2:

$$\gamma - \rho = \sqrt{-1}\, \partial\bar{\partial} f .$$

Now we look for another Kähler form $\omega' \in c$, i.e. we look for a smooth real function φ (also defined up to an additive constant), where

$$\omega' - \omega = \sqrt{-1}\, \partial\bar{\partial}\varphi$$

such that the Ricci form ρ' of ω' *coincides with* γ.

In other words, we want φ to satisfy

$$\rho' - \rho \equiv \sqrt{-1}\, \partial\bar{\partial} f ,$$

or equivalently, if g and g' are the Kähler metrics respectively associated with ω and ω',

$$\partial\bar{\partial}\{- \operatorname{Log} \det (g' g^{-1})\} \equiv \partial\bar{\partial} f$$

which immediately yields equation (1) with $\lambda = 0$:

$$- \operatorname{Log} \det (g' g^{-1}) = f ,$$

since anyway f is only defined up to an additive constant.

As ω and ω' are cohomologous and closed, so are the corresponding volume forms, therefore X has same volume measured with the metrics g and g'; this defines completely f, subject to the natural constraint mentioned above.

2. A Topological Lemma

In our setting, the continuity method becomes a "surjectivity method" since it is based on the following

LEMMA 2.1. *Let A, B be metric spaces, with $A \neq \emptyset$ and B connected. Let $P: A \to B$ be a continuous map. Assume:*

(i) *P is open,*

(ii) *P is proper, that is, for any compact subset K in $B, P^{-1}(K)$ is compact. Then P is surjective.*

Proof. We only need to prove that $P(A)$ is closed. Let b_0 be a point in $\overline{P(A)}$. Since B is a metric space, there exists a sequence $(b_i)_{i>0}$ in $P(A)$ converging to b_0. The subset $K = \{b_0, b_1, b_2, ...\}$ is compact, hence so is $PP^{-1}(K)$. The latter contains $b_1, ..., b_i, ...$, hence b_0, and it is obviously contained in $P(A)$. Q.E.D.

In order to make use of this lemma, we shall need some inverse function theorem for (i), and some *a priori* estimates for (ii).

3. LOCAL INVERSION

THEOREM 3.1. *Let X be a smooth compact manifold, V and W smooth vector bundles on X, U an open set in $C^\infty(X, V)$, and $P: U \to C^\infty(X, W)$, a smooth nonlinear elliptic partial differential operator. Let A and B be LCFC submanifolds of U and of $C^\infty(X, W)$ respectively, such that the restriction P_A of P to A, sends A into B. Then the Jacobian criterion holds for P_A, namely, if the derivative of $P_A: A \to B$ is invertible at $\varphi_0 \in A$, then P_A is a local diffeomorphism near φ_0.*

This is a convenient variant of the Nash-Moser theorem (e.g. [14]) regarding suitable restrictions of elliptic operators. It is established in a separate paper [11] (see also [22]). It relies only on the *classical* (Banach) inverse function theorem combined with *elliptic regularity.*

Remark 3.2. The Nash-Moser theorem has been studied by many authors, see the bibliography below and further references in [14] [15] [25].

4. PROPERNESS

In view of (2), theorem 3.1 implies that P_λ is open. We want to apply lemma 2.1 in order to prove that P_λ is surjective from A_λ to B_λ. Since $P_\lambda(A_\lambda) \neq \emptyset$ (it contains 0), and since B_λ is connected, this amounts to proving that P_λ is *proper.* Let us explain why *a priori* estimates imply properness.

Concerning subsets in A_λ we have

PROPOSITION 4.1. *A subset* S *in* A_λ *is relatively compact in* A_λ *iff its closure* $\bar{S}$ *in* $C^\infty(X)$ *lies inside* A_λ *and* S *is bounded in* $C^\infty(X)$.

This readily follows from Ascoli theorem which implies the well-known fact [12] (p. 231) that in $C^\infty(X)$ (and in any *closed* LCFC submanifold of $C^\infty(X)$, such as B_λ, as well) bounded subsets are relatively compact and vice-versa; hence, *compact* subset of A_λ are nothing but *bounded closed strictly interior* subsets of A_λ. Explicitely, let us state the

COROLLARY 4.2. *A closed subset* S *in* A_λ *is compact if and only if there exists a sequence* $(C_i), i \in \mathbf{N}$, *of positive numbers, such that for any* φ *in* S *the following estimates hold:*

$$\| (g')^{-1} \| =: \sup_X | (g')^{-1} | \leqslant C_0 ,$$

$$\forall i \in \mathbf{N} , \quad \| D^i \varphi \| =: \sup_X | D^i \varphi | \leqslant C_i ,$$

where $|\cdot|$ *denotes some natural norms of tensors in the original metric* g, *and* $D =: (\nabla, \bar{\nabla})$ *is the total covariant differentiation with respect to the metric* g.

Proof. Indeed S is closed and bounded. Moreover, since for $\varphi \in S$,

$$\| (g')^{-1} \| \leqslant C_0$$

all the eigenvalues of $(g')^{-1}$ (which are *positive*) are uniformly bounded *from above*, hence those of g' are uniformly bounded *from below*, in other words:

$$\exists \varepsilon > 0 , \quad \forall \varphi \in S , \quad g' \geqslant \varepsilon g ,$$

or equivalently $\bar{S}$ lies *strictly inside* A_λ. Q.E.D.

In the next sections we will show that if f belongs to some *compact* (i.e. bounded and closed) subset K of B_λ, defined by a sequence (K_i), $i \in \mathbf{N}$, such that $\| D^i f \| \leqslant K_i$, then for $\varphi \in A_\lambda$ satisfying $P_\lambda(\varphi) = f$, the following *a priori* estimates hold:

$$\| \varphi \| \leqslant C_0 , \quad \forall i \in \mathbf{N} , \quad \| D^i \nabla \bar{\nabla} \varphi \| \leqslant C_{i+2} .$$

These estimates imply that P_λ is *proper*, i.e. that $S = P_\lambda^{-1}(K)$ is compact, according to the following

PROPOSITION 4.3. *Let* S *be a closed subset in* A_λ. *Suppose that there exists a sequence* $(C_i), i \in \mathbf{N}$, *such that for any* φ *in* S, *the following estimates hold:*

$$\| \varphi \| \leqslant C_0 , \quad \| P_\lambda(\varphi) \| \leqslant C_0 , \quad \forall i \in \mathbf{N} , \quad \| D^i \nabla\bar{\nabla}\varphi \| \leqslant C_{i+2} .$$

Then S *is compact.*

Proof. The first two estimates imply a uniform estimate

$$| \operatorname{Log} \det (g' g^{-1}) | \leqslant E .$$

The estimate on $\| \nabla\bar{\nabla}\varphi \|$ yields another one:

$$\| g' \| \leqslant F .$$

These two estimates yield

$$\| (g')^{-1} \| \leqslant G .$$

Now from $\| D^i \nabla\bar{\nabla}\varphi \| \leqslant C_{i+2}$ we infer

$$\| D^i \Delta\varphi \| \leqslant \tilde{C}_{i+2}$$

since D and g^{-1} commute (Δ denotes the Laplacian in the metric g). As Δ performs a continuous linear automorphism of the Fréchet space of smooth functions *with zero average* (by Fredholm theory), the Closed Graph Theorem implies the missing estimates. Q.E.D.

Remark 4.4. Actually we have been considering two *gradings* of $C^\infty(X)$ [14]. The usual one, namely the one defined, $\forall u \in C^\infty(X)$, by

$$\| u \|_0 = \sup_X | u | ,$$
$$\| u \|_i = \| u_i \|_{i-1} + \| D^i u \| , \quad i \geqslant 1 ,$$

and another one, well-adapted here since the true unknown is a Kähler metric, defined by

$$\| u \|_0^* = \| u \|_0 , \quad \| u \|_1^* = \| u \|_1 ,$$
$$\| u \|_i^* = \| u \|_{i-1}^* + \| D^{i-2}(\nabla\bar{\nabla} u) \| , \quad i \geqslant 2 .$$

Although it is unnecessary for the purpose of proposition 4.3, it can be shown globally (without Schauder theory) that these two gradings are *tamely* equivalent [14] of degree 2 and base 0 [10] (section 5). Hence, they define the same topology.

5. A PRIORI ESTIMATES: THE ORIGINAL WAY

According to proposition 4.3 we must prove now that, given any sequence of positive real numbers (K_i), $i \in \mathbf{N}$, there exists a sequence (C_i) such that

$$\forall i \in \mathbf{N}\,, \quad \| D^i P_\lambda(\varphi) \| \leqslant K_i$$

implies

$$\| \varphi \| \leqslant C_0\,, \quad \forall i \in \mathbf{N}\,, \quad \| D^i \nabla \bar{\nabla} \varphi \| \leqslant C_{i+2}\,.$$

These are *a priori* estimates of order zero, two, three, and so on .... In case $\lambda > 0$, the C^0 estimate is straightforward [2]. In case $\lambda = 0$, it becomes very tricky; proofs simpler than Yau's original one [24] (p. 352-359), based on the idea of uniformly estimating the $L^p(dX_g)$ norms of φ, may be found in [16] (dimension 2), [3] [21] and [4] (p. 148-149).

Estimates of order two and three are carried out by means of tensor calculus and of the Maximum Principle (for elliptic equations) [20] applied to *suitable* test functions. Though it is not everywhere clear in [21] [24], it is worth noting that the computations can be written intrinsically, i.e. without any reference to a *particular* system of coordinates (e.g. [2]), or even *coordinate free* (see section 6 below).

Further regularity is then recovered by Schauder theory e.g. [5] (lemma 1). In the sequel, we show how further estimates can be carried out instead, *just going ahead with coordinate free tensor calculus.* This occurs actually for any fully nonlinear second order elliptic equation on a compact Riemannian manifold, via a straightforward imitation of the device below.

Remark 5.1. It follows from the C^2 *a priori* estimates that the metrics g' are *a priori* uniformly equivalent to the original metric g (see e.g. [3], p. 75).

6. COORDINATE FREE TENSOR CALCULUS

Even coordinate free tensor calculus needs indices. Usually these indices refer to a *local* frame. Another way is to view these indices *globally* as labelling copies of the holomorphic and antiholomorphic tangent and cotangent bundles. From this point of view, a tensor written with indices is a section of the tensor product of a family of bundles indexed by an *unordered* set of indices (disregarding those indices subject to the summation convention).

We extend the summation convention as follows: we will be concerned only with lower indices. If a letter occurs twice, it refers to a contraction, which is taken with respect to g or to g' according to whether the letter occurs with a bar or with a prime. So,

$$T_{\dots a \dots \bar{a} \dots} \text{ stands for } g^{a\bar{b}} T_{\dots a \dots \bar{b} \dots}, \quad \text{while}$$
$$T_{\dots a \dots a' \dots} \text{ stands for } g'^{a\bar{b}} T_{\dots a \dots \bar{b} \dots} \quad .$$

As usual if $T_{a \dots l}$ is a tensor, further lower indices refer to covariant differentiation (with respect to g); so,

$$T_{a \dots lm} \text{ stands for } \nabla_m T_{a \dots l}, \quad \text{while}$$
$$T_{a \dots l\bar{m}} \text{ stands for } \bar{\nabla}_{\bar{m}} T_{a \dots l} \quad .$$

Our indices will be latin letters; greek letters will denote multi-indices. If α is a multi-index, $\bar{\alpha}$ will denote the *conjugate* multi-index (for instance if $\alpha = a\bar{b}c$, then $\bar{\alpha} = \bar{a}b\bar{c}$), while $|\alpha|$ denotes its length. We shall say that α is *mixed* if its length is at least two and, among the first two letters, *exactly* one has a bar.

The notations D, ∇, $\bar{\nabla}$, $\| \ \|$, were introduced in section 4.

Remark 6.1. Since covariant differentiation (with respect to g) and contraction with respect to g' *do not* commute, we observe that, for instance, the difference (recall $g' = g + \nabla\bar{\nabla}\varphi$)

$$\varphi_{aa'\alpha b} - (\varphi_{aa'\alpha})_b \equiv \varphi_{ac\alpha}\,\varphi_{a'c'b} \tag{3}$$

does not vanish.

7. Higher order a priori estimates: generalities

We want to prove by induction,

Proposition 7.1. *Given* $n \geqslant 4$, *a sequence* (K_i), $i \in \mathbf{N}$, *and a finite sequence* $C_0, \dots, C_{n-1}$, *there exists* C_n *such that:*

$$\| \varphi \| \leqslant C_0, \quad \forall i = 0, \dots, n-3, \quad \| D^i \nabla\bar{\nabla}\varphi \| \leqslant C_{i+2}$$
$$\text{and} \quad \forall i \in \mathbf{N}, \quad \| D^i P_\lambda(\varphi) \| \leqslant K_i,$$

implies

$$\| D^{n-2} \nabla\bar{\nabla}\varphi \| \leqslant C_n .$$

Actually one needs $\| D^i P_\lambda(\varphi) \| \leqslant K_i$ *only for* $0 \leqslant i \leqslant n$, *hence* C_n *depends only upon* $(C_0, ..., C_{n-1}, K_0, ..., K_n)$.

Hereafter, by "a constant", we will mean a constant which depends only upon the given constants $(C_0, ..., C_{n-1}, K_0, ..., K_n)$.

Let us explain a further convention.

Convention 7.2. We will have to consider sums of tensors obtained via contractions of tensor polynomials in the variables $(g')^{-1}$, $\nabla\bar{\nabla}\varphi, ..., D^i\nabla\bar{\nabla}\varphi, ...$. The present convention helps describing the variables occuring in (still) uncontrolled expressions.

First of all, *given* $\varphi \in A_\lambda$ and an integer $n \geqslant 3$, we denote by E_{n-1} the (finite dimensional complex) vector space generated by all contracted tensor polynomials, with degree of homogeneity at most $2n$, in the variables

$$(g')^{-1}, \quad \nabla\bar{\nabla}\varphi, \quad D\nabla\bar{\nabla}\varphi, ..., D^{n-3}\nabla\bar{\nabla}\varphi, \quad D^i P_\lambda(\varphi), \quad i = 0, ..., n.$$

In order to prove 7.1, we will compute *modulo* E_{n-1}.

Given integers $p, ..., s$, all of them $\geqslant n$, we will say that mod. E_{n-1} a tensor T is "of the form $T_{p,...,s}$", whenever mod. E_{n-1} it is a sum of contractions of tensors

$$A \otimes D^{p-2}\nabla\bar{\nabla}\varphi \otimes ... \otimes D^{s-2}\nabla\bar{\nabla}\varphi,$$

where the A's are in E_{n-1}.

Furthermore for $s \geqslant n$, under the assumptions of 7.1, we will say that a *scalar* term $T_{s,s}$ is *coercive*, if for any other term of the form T'_s (resp. $T''_{s,s}$) there exists a constant C such that:

$$| T'_s | \leqslant C(T_{s,s})^{\frac{1}{2}} \quad (\text{resp. } | T''_{s,s} | \leqslant C T_{s,s}).$$

We present now three lemmas which illustrate the previous convention.

LEMMA 7.3. *Given integers* $s \geqslant n \geqslant 3$, *the covariant derivative (in metric* g*) of a term of the form* T_s mod. E_{n-1}, *is of the form* $(T_{s+1} + T_s)$ mod. E_n.

Proof. This is just because the derivative $D[(g')^{-1}]$ is a contracted tensor polynomial (of degree 3) in $(g')^{-1}$ and $D\nabla\bar{\nabla}\varphi$.

LEMMA 7.4. *If* $\dot{\alpha}$ *and* β *are two distinct mixed multi-indices of length* $(n+2)$ *obtained from each other by permutation, then the difference of covariant derivatives* $(\varphi_\alpha - \varphi_\beta)$ *is of the form* T_n mod. E_{n-1}.

Proof. On the Kähler manifold (X, g), commuting two consecutive covariant derivatives yields curvature terms only if the couple of derivatives concerned is *mixed* (for general commutation rules on Riemannian manifolds see e.g. [21], exposé XI, proposition 3.2). If so, say k and $\bar{l}$ are the permuted indices, the result will involve

$$R^{q}_{pk\bar{l}} \quad \text{(curvature tensor of } g)$$

with p and q *of the same type*. Explicitely:

$$\varphi_{\lambda k \bar{l} \mu} - \varphi_{\lambda \bar{l} k \mu} = \sum_{p} R_{p\bar{q}k\bar{l}}\, \varphi_{\nu q \tau}$$

for all p, ν, τ, such that $\nu p \tau \equiv \lambda\mu$. Hence the *types* of all the remaining non-permuted covariant derivatives $\varphi_{\nu q \tau}$ are *identically preserved.* In particular if γ and δ denote two multi-indices of length n obtained from each other by permutation, necessarily

$$(\varphi_{i\bar{j}\gamma} - \varphi_{i\bar{j}\delta}) \text{ is of the form } T_n \text{ mod. } E_{n-1},$$

since two *mixed* derivatives will keep bearing in first place on φ in the process of permutation.

The proof of lemma 7.4 is therefore reduced to the following two cases for the multi-indices α and β:

$$\begin{aligned} &\text{either} \quad \alpha = i\bar{j}k\lambda, \quad \beta = k\bar{j}i\lambda, \quad |\lambda| = n-1, \\ &\text{or} \quad \alpha = i\bar{j}k\bar{l}\mu, \quad \beta = k\bar{l}i\bar{j}\mu, \quad |\mu| = n-2. \end{aligned}$$

In the first case, one has identically on a Kähler manifold:

$$\varphi_\alpha - \varphi_\beta \equiv 0.$$

In the second case, the same reasoning as above holds for $(\varphi_\alpha - \varphi_\beta)$ since it can be written as

$$(\varphi_{i\bar{j}k\bar{l}\mu} - \varphi_{ik\bar{j}\bar{l}\mu}) + (\varphi_{ki\bar{l}\bar{j}\mu} - \varphi_{k\bar{l}i\bar{j}\mu}),$$

each of these two commutations being clearly of the form T_n mod. E_{n-1}.

Q.E.D.

Remark 7.5. The fact that commutation formulae involve only *mixed* derivatives was already a crucial detail in the proofs of the second and third order *a priori* estimates.

LEMMA 7.6. *The tensor* $\varphi_{aa'\alpha}$ *where* α *is a mixed multi-index of length* n *is,* mod. E_{n-1}, *of the form:*

$$
\begin{array}{lll}
T_{3,3} + T_2 & \textit{when} & n = 2 , \\
T_{4,3} + T_{3,3,3} + T_3 & \textit{when} & n = 3 , \\
T_5 + T_{4,4} + T_4 & \textit{when} & n = 4 , \\
T_{n+1} + T_n & \textit{when} & n \geqslant 5 .
\end{array}
$$

Proof. The cases $n = 2, 3, 4, 5$, must be checked bare-handed. There is no difficulty. Then, for $n \geqslant 5$, one can proceed by induction on n. Indeed assume,

$$\varphi_{aa'\alpha} = T_{n+1} + T_n \text{ mod. } E_{n-1} , \quad \text{for some } n = |\alpha| \geqslant 5 .$$

Recall formula (3) and lemma 7.3; differentiating once the above equality yields

$$\varphi_{aa'\alpha b} = (T_{n+1} + T_n)_b + \varphi_{ac\alpha}\,\varphi_{a'c'b} = T_{n+2} + T_{n+1} \text{ mod. } E_n ,$$

since $|ac\alpha| = n + 2$. The same is true with $\bar{b}$ instead of b. Q.E.D.

Remark 7.7. The preceding lemma offers a perspective which brings some light on the type of difficulties to be expected for carrying out *a priori* estimates of each order. In particular, one may anticipate that a special step should be required for $n = 4$ (in order to kill the effect of the term $T_{4,4}$) and that the same (simpler) procedure should then apply, arguing by iteration, for any $n \geqslant 5$.

Notice also that the hardest case appears to be $n = 3$. Indeed, following Calabi [8] one must guess the very special *coercive* functional [1] [24]

$$S_{3,3} = \varphi_{ab'c}\,\varphi_{a'bc'} ,$$

perform a careful calculation of $\Delta'(S_{3,3})$ and use either the Maximum Principle [24] or a recurrence on $L^p(dX_{g'})$ norms of $S_{3,3}$ [1]. The *approximate* tensor calculus which we may conveniently use hereafter would not be effective for the case $n = 3$.

8. A PRIORI ESTIMATES OF ORDER FOUR

In order to prove 7.1 with $n = 4$, we consider the functional:

$$S_{4,4} = \varphi_{a\bar{b}cd}\,\varphi_{\bar{a}b\bar{c}\bar{d}} + \varphi_{a\bar{b}\bar{c}d}\,\varphi_{\bar{a}bc\bar{d}} .$$

It is enough to estimate $S_{4,4}$ since it is *coercive.* Let us compute $-\Delta'(S_{4,4})$. One readily obtains:

$$-\Delta'(S_{4,4}) = T_{6,4} + T_{5,5} \quad (\text{mod. } E_3) ,$$

where $T_{5,5}$ is *coercive*, while the sixth order derivatives in $T_{6,4}$ occur through $\varphi_{a\bar{b}\alpha c c'}$ with $|\alpha| = 2$.

In view of 7.4 and 7.6, after bringing the indices cc' in first position, we get

$$(4) \qquad -\Delta'(S_{4,4}) = T_{5,5} + T_{5,4} + T_{4,4,4} + T_{4,4} + T_4 \qquad (\text{mod. } E_3)$$

where $T_{5,5}$ is the *coercive* term from above.

As expected in remark 7.7, in order to control the term $T_{4,4,4}$, we need to consider instead of $S_{4,4}$ another functional, namely:

$$\theta = S_{4,4} \exp(\varepsilon\, \varphi_{a\bar{b}c}\, \varphi_{\bar{a}b\bar{c}}),$$

where ε is a constant to be chosen later on. Then we compute the quantity

$$Q = -(\Delta'\theta) \exp(-\varepsilon \varphi_{a\bar{b}c}\, \varphi_{\bar{a}b\bar{c}});$$

and we easily find

$$Q = -\Delta'(S_{4,4}) + \varepsilon T_{4,4,4,4} + \varepsilon^2 T'_{4,4,4,4} + \varepsilon T_{5,4,4} \qquad (\text{mod. } E_3),$$

where $T'_{4,4,4,4}$ is a square and where

$$T_{4,4,4,4} = S_{4,4}(\varphi_{a\bar{b}cd}\, \varphi_{\bar{a}b\bar{c}d'} + \varphi_{a\bar{b}cd'}\, \varphi_{\bar{a}b\bar{c}d}).$$

So there exists a constant c_1 such that (see remark 5.1),

$$(S_{4,4})^2 \leqslant c_1\, T_{4,4,4,4}.$$

Furthermore we may choose constants c_i such that,

$$|T_{5,4,4}| \leqslant c_2\, S_{4,4}(T_{5,5})^{\frac{1}{2}}, \quad |T_{5,4}| \leqslant c_3(T_{5,5}\, S_{4,4})^{\frac{1}{2}},$$
$$|T_{4,4,4}| \leqslant c_4(S_{4,4})^{\frac{3}{2}}, \quad |T_{4,4}| \leqslant c_5 S_{4,4}, \quad |T_4| \leqslant c_6(S_{4,4})^{\frac{1}{2}}.$$

By splitting $T_{5,5}$ in its two halves and by putting each half together with $T_{5,4,4}$ and $T_{5,4}$ respectively, one obtains:

$$Q \geqslant \left(\frac{\varepsilon}{c_1} - \frac{1}{2}\varepsilon^2 c_2^2\right)(S_{4,4})^2 - c_4(S_{4,4})^{\frac{3}{2}} - \left(c_5 + \frac{1}{2}c_3^2\right) S_{4,4} - c_6(S_{4,4})^{\frac{1}{2}}.$$

Now ε must be chosen small enough in order for the coefficient of $(S_{4,4})^2$ to be *strictly* positive: $\varepsilon \in (0, (2/c_1\, c_2^2))$.

To complete the proof, one argues that $Q(z_0) \leqslant 0$ at a point $z_0 \in X$ where θ assumes its *maximum* on X, which implies

$$S_{4,4}(z_0) \leqslant c_7,$$

for some controlled constant c_7, and anywhere else on X, since $\theta \leqslant \theta(z_0)$ and $\| D\nabla\bar{\nabla}\varphi \| \leqslant C_3$, one infers that:

$$S_{4,4} \leqslant c_7 \exp(2\varepsilon C_3).$$

9. A PRIORI ESTIMATES OF ORDER FIVE AND MORE

Here, in order to prove 7.1 with $n \geqslant 5$, we consider the functional:

$$S_{n,n} = \frac{1}{2} \sum_{|\alpha| = n-2} \varphi_{a\bar{b}\alpha}\, \varphi_{\bar{a}b\bar{\alpha}}$$

(the coefficient $\frac{1}{2}$ appears for both definitions of $S_{4,4}$ to agree).

Again $S_{n,n}$ is *coercive* and we compute in a similar way,

$$-\Delta'(S_{n,n}) = T_{n+2,n} + T_{n+1,n+1} \quad (\text{mod. } E_{n-1}),$$

where $T_{n+1,n+1}$ is *coercive.* As for $T_{n+2,n}$, proceeding as in the previous section, we find:

$$T_{n+2,n} = T_{n+1,n} + T_{n,n} + T_n \quad (\text{mod. } E_{n-1}).$$

Hence,

$$-\Delta'(S_{n,n}) = T_{n+1,n+1} + T_{n+1,n} + T_{n,n} + T_n \quad (\text{mod. } E_{n-1}),$$

with $T_{n+1,n+1}$ coercive. Changing n into $(n-1)$, for $n \geqslant 6$, yields *still modulo* E_{n-1}

$$-\Delta'(S_{n-1,n-1}) = T'_{n,n} + T'_n \quad (\text{mod. } E_{n-1}).$$

In view of formula (4) of the preceding section, this holds for $n = 5$ as well. From the *coercivity* of $T'_{n,n}$ we may choose constants $c_i > 0$, such that

$$-\Delta'(S_{n-1,n-1}) \geqslant c_1 S_{n,n} - c_2(S_{n,n})^{\frac{1}{2}} - c_3.$$

Moreover we may choose constants c_i such that

$$| T_{n+1,n} | \leqslant 2c_4(T_{n+1,n+1} S_{n,n})^{\frac{1}{2}}, \quad | T_{n,n} | \leqslant c_5 S_{n,n}, \quad | T_n | \leqslant c_6(S_{n,n})^{\frac{1}{2}},$$
$$\text{and} \quad c_1 c_7 > c_4^2 + c_5.$$

We obtain,

$$-\Delta'(S_{n,n}+c_7S_{n-1,n-1}) \geqslant (c_1c_7-c_4^2-c_5)S_{n,n} - (c_6+c_2c_7)(S_{n,n})^{\frac{1}{2}} - c_3c_7$$

and the proof may be easily completed.

10. The analytic point of view

Since equation (1) is *elliptic* and g, as a Kähler metric, is real analytic for the underlying real (analytic) structure of X, by the general elliptic regularity theory e.g. [17], p. 266-277 if $P_\lambda(\varphi)$ is real analytic so are φ and g'. Hence a purely analytic proof would be desirable.

Real analytic inverse function theorems are available since the work of J. Nash [19] who made a decisive use of smoothing operators (see also [13]). A theorem of H. Jaccbowitz [15] (p. 203) (see also [25], p. 94-101, 137-138) is available, the proof of which is purely analytical and does not use smoothing operators. This approach was first initiated by A. Kolmogorov (1954) and developed by V. Arnold (1961) (see references in [18]), and by J. Moser [18] (p. 513-533). Unfortunately, the application to nonlinear elliptic operators is not achieved.

A further trouble arises from the fact that the space of analytic functions *is not metrizable.*

Last but not least, we could not carry out analytic *a priori* estimates.

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