

# 8. A PRIORI ESTIMATES OF ORDER FOUR

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.09.2024**

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

$$\begin{aligned}
T_{3,3} + T_2 & \quad \text{when } n = 2, \\
T_{4,3} + T_{3,3,3} + T_3 & \quad \text{when } n = 3, \\
T_5 + T_{4,4} + T_4 & \quad \text{when } n = 4, \\
T_{n+1} + T_n & \quad \text{when } n \geq 5.
\end{aligned}$$

*Proof.* The cases  $n = 2, 3, 4, 5$ , must be checked bare-handed. There is no difficulty. Then, for  $n \geq 5$ , one can proceed by induction on  $n$ . Indeed assume,

$$\Phi_{aa'\alpha} = T_{n+1} + T_n \text{ mod. } E_{n-1}, \quad \text{for some } n = |\alpha| \geq 5.$$

Recall formula (3) and lemma 7.3; differentiating once the above equality yields

$$\Phi_{aa'\alpha b} = (T_{n+1} + T_n)_b + \Phi_{ac\alpha} \Phi_{a'c'b} = T_{n+2} + T_{n+1} \text{ mod. } E_n,$$

since  $|ac\alpha| = n + 2$ . The same is true with  $\bar{b}$  instead of  $b$ . Q.E.D.

*Remark 7.7.* The preceding lemma offers a perspective which brings some light on the type of difficulties to be expected for carrying out *a priori* estimates of each order. In particular, one may anticipate that a special step should be required for  $n = 4$  (in order to kill the effect of the term  $T_{4,4}$ ) and that the same (simpler) procedure should then apply, arguing by iteration, for any  $n \geq 5$ .

Notice also that the hardest case appears to be  $n = 3$ . Indeed, following Calabi [8] one must guess the very special *coercive* functional [1] [24]

$$S_{3,3} = \Phi_{ab'c} \Phi_{a'bc'},$$

perform a careful calculation of  $\Delta'(S_{3,3})$  and use either the Maximum Principle [24] or a recurrence on  $L^p(dX_{g'})$  norms of  $S_{3,3}$  [1]. The *approximate* tensor calculus which we may conveniently use hereafter would not be effective for the case  $n = 3$ .

## 8. A PRIORI ESTIMATES OF ORDER FOUR

In order to prove 7.1 with  $n = 4$ , we consider the functional:

$$S_{4,4} = \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}} \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}} + \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}} \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}}.$$

It is enough to estimate  $S_{4,4}$  since it is *coercive*. Let us compute  $-\Delta'(S_{4,4})$ . One readily obtains:

$$-\Delta'(S_{4,4}) = T_{6,4} + T_{5,5} \quad (\text{mod. } E_3),$$

where  $T_{5,5}$  is *coercive*, while the sixth order derivatives in  $T_{6,4}$  occur through  $\varphi_{\bar{a}\bar{b}\alpha c c'}$  with  $|\alpha| = 2$ .

In view of 7.4 and 7.6, after bringing the indices  $cc'$  in first position, we get

$$(4) \quad -\Delta'(S_{4,4}) = T_{5,5} + T_{5,4} + T_{4,4,4} + T_{4,4} + T_4 \pmod{E_3}$$

where  $T_{5,5}$  is the *coercive* term from above.

As expected in remark 7.7, in order to control the term  $T_{4,4,4}$ , we need to consider instead of  $S_{4,4}$  another functional, namely:

$$\theta = S_{4,4} \exp(\varepsilon \varphi_{\bar{a}\bar{b}c} \varphi_{\bar{a}\bar{b}\bar{c}}),$$

where  $\varepsilon$  is a constant to be chosen later on. Then we compute the quantity

$$Q = -(\Delta'\theta) \exp(-\varepsilon \varphi_{\bar{a}\bar{b}c} \varphi_{\bar{a}\bar{b}\bar{c}});$$

and we easily find

$$Q = -\Delta'(S_{4,4}) + \varepsilon T_{4,4,4,4} + \varepsilon^2 T'_{4,4,4,4} + \varepsilon T_{5,4,4} \pmod{E_3},$$

where  $T'_{4,4,4,4}$  is a square and where

$$T_{4,4,4,4} = S_{4,4}(\varphi_{\bar{a}\bar{b}c d} \varphi_{\bar{a}\bar{b}\bar{c}d'} + \varphi_{\bar{a}\bar{b}c d'} \varphi_{\bar{a}\bar{b}\bar{c}d}).$$

So there exists a constant  $c_1$  such that (see remark 5.1),

$$(S_{4,4})^2 \leq c_1 T_{4,4,4,4}.$$

Furthermore we may choose constants  $c_i$  such that,

$$\begin{aligned} |T_{5,4,4}| &\leq c_2 S_{4,4} (T_{5,5})^{\frac{1}{2}}, & |T_{5,4}| &\leq c_3 (T_{5,5} S_{4,4})^{\frac{1}{2}}, \\ |T_{4,4,4}| &\leq c_4 (S_{4,4})^{\frac{3}{2}}, & |T_{4,4}| &\leq c_5 S_{4,4}, & |T_4| &\leq c_6 (S_{4,4})^{\frac{1}{2}}. \end{aligned}$$

By splitting  $T_{5,5}$  in its two halves and by putting each half together with  $T_{5,4,4}$  and  $T_{5,4}$  respectively, one obtains:

$$Q \geq \left( \frac{\varepsilon}{c_1} - \frac{1}{2} \varepsilon^2 c_2^2 \right) (S_{4,4})^2 - c_4 (S_{4,4})^{\frac{3}{2}} - \left( c_5 + \frac{1}{2} c_3^2 \right) S_{4,4} - c_6 (S_{4,4})^{\frac{1}{2}}.$$

Now  $\varepsilon$  must be chosen small enough in order for the coefficient of  $(S_{4,4})^2$  to be *strictly* positive:  $\varepsilon \in (0, (2/c_1 c_2^2))$ .

To complete the proof, one argues that  $Q(z_0) \leq 0$  at a point  $z_0 \in X$  where  $\theta$  assumes its *maximum* on  $X$ , which implies

$$S_{4,4}(z_0) \leq c_7,$$

for some controlled constant  $c_7$ , and anywhere else on  $X$ , since  $\theta \leq \theta(z_0)$  and  $\|D\bar{\nabla}\bar{\nabla}\varphi\| \leq C_3$ , one infers that:

$$S_{4,4} \leq c_7 \exp(2\varepsilon C_3).$$

### 9. A PRIORI ESTIMATES OF ORDER FIVE AND MORE

Here, in order to prove 7.1 with  $n \geq 5$ , we consider the functional:

$$S_{n,n} = \frac{1}{2} \sum_{|\alpha|=n-2} \varphi_{a\bar{b}\alpha} \varphi_{\bar{a}b\bar{\alpha}}$$

(the coefficient  $\frac{1}{2}$  appears for both definitions of  $S_{4,4}$  to agree).

Again  $S_{n,n}$  is *coercive* and we compute in a similar way,

$$-\Delta'(S_{n,n}) = T_{n+2,n} + T_{n+1,n+1} \pmod{E_{n-1}},$$

where  $T_{n+1,n+1}$  is *coercive*. As for  $T_{n+2,n}$ , proceeding as in the previous section, we find:

$$T_{n+2,n} = T_{n+1,n} + T_{n,n} + T_n \pmod{E_{n-1}}.$$

Hence,

$$-\Delta'(S_{n,n}) = T_{n+1,n+1} + T_{n+1,n} + T_{n,n} + T_n \pmod{E_{n-1}},$$

with  $T_{n+1,n+1}$  *coercive*. Changing  $n$  into  $(n-1)$ , for  $n \geq 6$ , yields *still modulo*  $E_{n-1}$

$$-\Delta'(S_{n-1,n-1}) = T'_{n,n} + T'_n \pmod{E_{n-1}}.$$

In view of formula (4) of the preceding section, this holds for  $n = 5$  as well. From the *coercivity* of  $T'_{n,n}$  we may choose constants  $c_i > 0$ , such that

$$-\Delta'(S_{n-1,n-1}) \geq c_1 S_{n,n} - c_2 (S_{n,n})^{\frac{1}{2}} - c_3.$$

Moreover we may choose constants  $c_i$  such that

$$|T_{n+1,n}| \leq 2c_4 (T_{n+1,n+1} S_{n,n})^{\frac{1}{2}}, \quad |T_{n,n}| \leq c_5 S_{n,n}, \quad |T_n| \leq c_6 (S_{n,n})^{\frac{1}{2}},$$

and  $c_1 c_7 > c_4^2 + c_5$ .