§8. Appendix : Real Forms and the generalized bruhat decomposition

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§ 8. Appendix: Real Forms and the Generalized Bruhat Decomposition

Let $G_{\mathbf{C}}$ be a reductive complex algebraic group, as usual, and let $P = P_I$, $Q = P_J$ be parabolic subgroups. Let $H_{\mathbf{C}}$ be "the" Levi factor of P with maximal compact subgroup H. Explicitly, $H_{\mathbf{C}}$ is the (closed, connected) subgroup whose Lie algebra is generated by $\mathfrak{t}_{\mathbf{C}}$ and the $X_{-\alpha_{\pm\alpha}}$, $\alpha \in I$. We have $P = H_{\mathbf{C}} U_I$, where the unipotent radical U_I corresponds to the positive roots not in the span of I.

(8.1) Theorem. *The P-orbits in $G_{\mathbf{C}}/Q$ are holomorphic vector bundles over fiag varieties of $H_{\mathbf{C}}$.*

Theorem (8.1) is certainly well known, although not so easy to find in the literature. In this section we will prove (8.1) and its loop group analogue in a more explicit form, and show how one may easily deduce the Bruhat decomposition for real forms from this. (The proofs of various technical lemmas about root systems will be omitted. The details are somewhat tedious, but not difficult.)

(8.2) Lemma. *Each $W_I - W_J$ double coset in W contains a unique element w of minimal length. For such a w we have*

(a) $\{x \in W_I : w^{-1}xw \in W_J\} = W_K$, *where* $K = \{s \in I : w^{-1}sw \in J\}$.

(b) *each $x \in W_I\, w\, W_J$ has a unique factorization of the form $x = vwy$, with $v \in (W_I)^K$, $y \in W_J$. Furthermore $l(vwy) = l(v) + l(w) + l(y)$, (in particular $vw \in W^J$).* □

Let w be minimal as in (8.2), and let $E = \{h \in H_{\mathbf{C}} : w^{-1}\, h\, w \in Q\}$ (i.e., E is the isotropy group of wQ in $H_{\mathbf{C}}$).

(8.3) Lemma. *E is a parabolic subgroup of $H_{\mathbf{C}}$, and its Levi factor $F_{\mathbf{C}}$ normalizes U_w.* □

We recall here that $U_w = \{u \in U : w^{-1}uw \in U^-\}$. In the present situation it is easy to see that $U_w \leqslant U_I$, and $w^{-1}U_w w \leqslant U_I^-$. The proof of (8.3) then reduces to a simple calculation in the root system. Now form the balanced product $H_{\mathbf{C}} \times_E U_w$, where E acts on U_w via the projection $E \to F_{\mathbf{C}}$. Since $\exp : u_w \to U_w$ is an ad-equivariant isomorphism of varieties,

$H_{\mathbf{C}} \times_E U_w$ is an algebraic vector bundle over the flag variety $H_{\mathbf{C}}/E$. Of course we also have $H_{\mathbf{C}}/E = H/F$ and $H_{\mathbf{C}} \times_E U_w = H \times_F U_w$ (by the Iwasawa decomposition).

(8.4) THEOREM. *The map* $\varphi: H_{\mathbf{C}} \times_E U_w \to PwQ/Q$ *given by* $(h, u) \mapsto huwQ/Q$ *is an isomorphism of varieties.*

Proof. Clearly φ is well—defined and surjective. To see that φ is injective, note that the Bruhat decomposition of $H_{\mathbf{C}}/E$ lifts to a cell decomposition of $H_{\mathbf{C}} \times_E U_w$; the cells are of the form $(U_\nu \nu) \times (U_w w)$, where ν ranges over $(W_I)^K$. By Lemma (8.2), the νw are distinct elements of W^J, so φ maps cells to Bruhat cells. Finally, (8.2) and the Steinberg lemma show that φ is injective on each cell. □

Example. Let $G_{\mathbf{C}} = GL(n, \mathbf{C})$, so $W = \sum_n$ and $S = (s_1, \cdots s_{n-1})$ as usual. Take $I = J = S - \{s_k\}$, so $G_{\mathbf{C}}/Q$ is the Grassman manifold of k-planes in n-space, and $W_I = \sum_k \times \sum_{n-k}$. The $(I-I)$-minimal elements are precisely the shuffles σ_i defined by $\sigma_i(r) = r$ (if $1 \leqslant r \leqslant i$), $\sigma_i(r) = k + r - i (i+1 \leqslant r \leqslant k)$; here $i \leqslant k$ and $k - i \leqslant n - k$. Note σ_i has length $(k-i)^2$. The $P(=Q)$-orbit of σ_i is $\{W \in G_{n,k}: \dim W \cap \mathbf{C}^k = i\}$, where $\mathbf{C}^k$ is the span of the first k basis vectors. This orbit can then be identified with the vector bundle $\hom(\gamma_{n-k,k-i}, \gamma^{\perp}_{k,i})$ over $G_{k,i} \times G_{n-k,k-i}$ (γ denoting the canonical bundle).

Now suppose given an involution σ on $G_{\mathbf{C}}$ (in normal form) with $G_{\mathbf{R}} = (G_{\mathbf{C}})^\sigma$, etc. We take I corresponding to the black nodes on the Satake diagram—i.e., I corresponds to the simple roots α such that $\sigma\alpha = -\alpha$. Also take $I = J$, so $Q = P_I$. We have $B_{\mathbf{R}} = Q^\sigma$ (by definition) and $W_{\mathbf{R}} = W^\sigma/W_I$ (W_I is usually denoted W_0). Note that σ preserves Q and hence permutes the $Q - Q$ double cosets.

(8.5) THEOREM.

(a) *If* $w \in W^\sigma \cap W^I$, *then* $QwQ = BwQ = U_w wQ$ *and* σ *acts on* U_w *as a conjugate linear involution.*

(b) *If* $w \notin W^\sigma$, $QwQ \cap G_{\mathbf{R}}$ *is empty.*

(8.6) COROLLARY (Bruhat decomposition). $G_{\mathbf{R}} = \coprod_{w \in W_{\mathbf{R}}} B_{\mathbf{R}} w B_{\mathbf{R}}$.

Proof. Note that on $H_{\mathbf{C}}/C(H_{\mathbf{C}})$, σ is the compact involution. In particular σ is the identity on $H/C(H)$; in fact $H = (C_K t_m)(C(H))$. It follows that $hwQ = wQ$ for $h \in H$, and hence $QwQ = UwQ = U_w wQ$. A calculation with roots shows that $\sigma(U_w) = U_w$, and (a) follows. Now consider a fixed w

of minimal length in $W_I w W_I$. To prove (b), we may as well assume that σ preserves QwQ; i.e., $\sigma(w) = awb$ with $a, b \in W_I$. We identify the orbit QwQ with $H \times_F U_w$ as in Theorem (8.4).

(8.7) LEMMA.

(a) $a \in N_H F$, *and* a *is well defined* mod F.

(b) $a^{-1}\sigma(U_w)\, a = U_w$. □

It now follows that σ acts as a conjugate linear bundle automorphism:

$$\sigma(huwQ) = hz\sigma(u)awQ = hzau'Q = hazu'Q,$$

where $u' = a^{-1}\sigma(u)a$ and $z \in C(H)$. Furthermore the action on the base H/F is given by $\sigma(hF) = haF$. Hence either σ acts freely on the base, and hence freely on the orbit, or else $a \in F$. In the latter case $\sigma(wW_I) = wW_I$. But it is a (trivial) exercise in linear algebra to show that this implies $\sigma(w) = w$. □

Example. Consider the involution $\sigma(A) = J\bar{A}J^{-1}$ on $SU(4)$, as in § 6. Q is the stabilizer of the standard 2-plane in $\mathbf{C}^4$ and $H = S(U(2) \times U(2))$. The minimal length elements (with respect to $I - I$, where $I = (s_1, s_3)$) are $1, s_2, s_2 s_1 s_3 s_2$ (shuffles as in the preceding example). The corresponding Q-orbits are respectively a point, a line bundle over $\mathbf{C}P^1 \times \mathbf{C}P^1$, and a cell of complex dimension four. The action of σ on $\mathbf{C}P^1 \times \mathbf{C}P^1$ is the obvious one on each factor, arising from the quaternionic "j" acting on complex lines in $\mathbf{H}$, and obviously is free. Taking fixed points yields the usual cell decomposition of the 4-sphere $\mathbf{H}P^1$.

Now consider our algebraic loop group $\tilde{G}_{\mathbf{C}}$. For simplicity we consider only parabolics P_I, P_J with $I, J \leqslant S$ (i.e. $P_I, P_J \leqslant P = G_{\mathbf{C}[z]}$), although this is not really necessary.

(8.8) THEOREM. *The P_I-orbits in $\tilde{G}_{\mathbf{C}}/P_J$ are holomorphic vector bundles over flag varieties of $H_{\mathbf{C}}$.* □

Here we note that although our notation is slightly ambiguous—in (8.8) P_I is a parabolic subgroup of $\tilde{G}_{\mathbf{C}}$, but it could also denote a parabolic in $G_{\mathbf{C}}$—the Levi factor $H_{\mathbf{C}}$ is the same for either interpretation. In any case the proof is identical to the proof of the classical case, with the affine root system replacing the ordinary root system. In particular the analogues of (8.2), (8.3), and (8.4) hold.

Example. Consider the P-orbits of $\tilde{G}_{\mathbf{C}}/P = \Omega_{alg}G$ (G simply-connected). These are indexed by homomorphisms $\lambda: S^1 \to T$ that lie in the closure of the dominant Weyl chamber ($\alpha(\lambda) \geqslant 0$ for all $\alpha \in \Phi^+$), and are precisely the stable manifolds of the energy flow on ΩG ([28]). The Levi factor $H_{\mathbf{C}}$ is just $G_{\mathbf{C}}$ in this case, so $H = G$, and $P\lambda P/P$ is a vector bundle over $G/C_G\lambda$. Now $W\lambda W = {}_{\lambda' \sim \lambda}\lambda' W$, where $\sim$ means W-conjugate. Hence, although λ will not be minimal in $W\lambda W$, the formula of Iwahori and Matsunoto shows that the minimal element has the form λw, and has length $\sum_{\alpha \in \Phi^+} \alpha(\lambda) - |\{\alpha > 0: \alpha(\lambda) \neq 0\}|$. Hence this length is the complex dimension of the vector bundle in question, and one can even determine the bundle explicitly. For example, suppose $G = SU(n)$. Then λ corresponds to a sequence of integers $(b_1, ..., b_n)$ with $\sum b_i = 0$ and $b_1 \geqslant b_2 ... \geqslant b_n$. Write this sequence in the form $(a_1, a_1, ... a_r, a_r)$, where there are i_1 entries a_1, i_2 entries a_2, etc. Then $G/C_G\lambda$ is the flag variety $U(n)/U(i_1) \times ... \times U(i_r)$. Over this there are r canonical bundles ξ_k of dimension i_k; let $\xi_{kl} = \hom(\xi_l, \xi_k)$. Now $\sum_{\alpha > 0} \alpha(\lambda) = \sum_{k<l} (a_k - a_l)\, i_k i_l$, and $|(\alpha > 0: \alpha(\lambda) \neq 0)| = \sum_{k<l} i_k i_l$. This suggests that the bundle is $\oplus_{k<l} (a_k - a_l - 1)\, \xi_{kl}$, and indeed this is easily verified. For the ξ_{kl} are precisely the irreducible components of the adjoint action of $C_G\lambda$ on the Lie algebra of the unipotent radical of the corresponding parabolic (i.e. the Lie algebra spanned by the X_α with $\alpha > 0, \alpha(\lambda) \neq 0$). Then one can check that $U_{\lambda w}$ corresponds to the strictly positive roots (n, α) (i.e. $n \geqslant 1$) such that $\lambda^{-1} \cdot (n, \alpha) = (n - \alpha(\lambda), \alpha)$ is strictly negative (i.e. $n - \alpha(\lambda) \leqslant -1$). Furthermore since $C_G\lambda$ consists of constant loops, it preserves the sum of the root subalgebras of fixed height n. Hence each ξ_{kl} (thought of as a representation of $C_G\lambda$) occurs in $U_{\lambda w}$ with multiplicity $a_k - a_l - 1$, which proves our assertion.

Finally, consider the involution t on $\tilde{G}_{\mathbf{C}}$. Theorem (8.5) and its proof carry over without difficulty, and we obtain:

(8.9) THEOREM. $\tilde{G}_{\mathbf{R}} = \coprod_{w \in W_{\mathbf{R}}} \tilde{B}_{\mathbf{R}}\, w \tilde{B}_{\mathbf{R}}$.