

# §3. Algebraic lemmas

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where  $r = (m-1)/2$  and  $\varepsilon(i) = (-1)^{i+1}$ . It is easy to show that under a different choice of natural bases and bases  $h_0, h_1, \dots, h_m$  the element  $d$  is replaced by  $\pm gq\bar{q}d$  with  $g \in G, q \in Q \setminus 0$ . Thus the set  $\{\pm gq\bar{q}d \mid g \in Q \setminus 0\} \subset Q$  does not depend on the choice of bases. It also does not depend on the choice of triangulation in  $M$ . It is this set which is  $\omega(M)$ .

An explicit formula established in [4] enables us to calculate  $\omega(M)$  in terms of the orders of  $\mathbf{Z}[G]$ -modules  $H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; \mathbf{Z}), H_*(\tilde{M}) = H_*(\tilde{M}; \mathbf{Z})$  and related modules. (The notion of the order of a module is recalled in Sec. 3.1.) Denote by  $J$  the image of the inclusion homomorphism  $H_r(\partial\tilde{M}) \rightarrow H_r(\tilde{M})$  where  $r = (m-1)/2$ . Then up to multiples of type  $q\bar{q}$  with  $q \in Q \setminus 0$

$$(4) \quad \omega(M) = \text{ord}(\text{Tors}_{\mathbf{Z}[G]} H_r(M, \partial M)) (\text{ord } J)^{\varepsilon(r)} \prod_{i=0}^{r-1} [\text{ord } H_i(\partial M)]^{\varepsilon(i)}$$

(see [4, Theorem 5.1.1]). Note that the equalities  $Q \otimes_{\mathbf{Z}[G]} H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; Q) = 0$  imply that  $H_*(\partial\tilde{M})$  and  $J$  are torsion  $\mathbf{Z}[G]$ -modules. Therefore  $\text{ord } H_i(\partial\tilde{M})$  and  $\text{ord } J$  are non-zero elements of  $\mathbf{Z}[G]$ .

We shall apply formula (4) in the case where  $M$  is the exterior of an  $n$ -component link  $K \subset S^m$  with odd  $m$ . The condition  $H_*(\partial M; Q) = 0$  is always fulfilled in this case. Here the field  $Q$  is canonically identified with the field of rational functions of  $n$  variables  $Q_n = Q(t_1, \dots, t_n)$ . Thus  $\omega(M) \subset Q_n$ . If  $m \geq 5$  then (4) implies that

$$\Delta(K)(t_1, \dots, t_n) \cdot \prod_{i=1}^n (t_i - 1) \subset \omega(M).$$

If  $m = 3$  then there exists a unique subset  $\alpha = \alpha(K)$  of the set  $\{1, 2, \dots, n\}$  such that

$$\Delta_{u(K)}(K)(t_1, \dots, t_n) \cdot \prod_{i \in \alpha} (t_i - 1) \subset \omega(M).$$

For proofs and details consult [4, § 5].

### § 3. ALGEBRAIC LEMMAS

3.1. PRELIMINARY DEFINITIONS. For a finitely generated module  $H$  over a (commutative) domain  $R$  we denote by  $\text{rk}_R H$  or, briefly, by  $\text{rk } H$  the integer  $\dim_Q(Q \otimes_R H)$  where  $Q = Q(R)$  denotes the field of fractions of  $R$ . For a  $R$ -linear homomorphism  $f: H \rightarrow H'$  we put  $\text{rk } f = \text{rk}_R f(H)$ . Note that if  $\bar{R}$  is the localization of  $R$  at some multiplicative system then  $Q(\bar{R}) = Q(R)$  and therefore the (exact) functor  $(H \mapsto \bar{R} \otimes_R H, f \mapsto \text{id}_{\bar{R}} \otimes f)$

preserves the ranks of modules and homomorphisms. If  $H, H'$  are finitely generated free  $R$ -modules and if  $A$  is the matrix of a  $R$ -homomorphism  $H \rightarrow H'$  with respect to some bases then  $\text{rk } f = \text{rk } A$  where  $\text{rk } A$  is the maximal integer  $r$  such that some  $r \times r$ -minor of  $A$  is non-zero.

If  $R$  is a unique factorization domain with 1 and if  $A$  is a matrix with  $n < \infty$  columns and possibly infinite number of rows then  $\Delta_i(A)$  denotes the greatest common divisor of the  $(n-i+1) \times (n-i+1)$ -minors of  $A$ . Here  $i = 1, 2, \dots$  and  $\Delta_i(A)$  is an element of  $R$  defined up to a unit multiple. If  $H$  is a finitely generated module over  $R$  and  $A$  is a presentation matrix of  $H$  then  $\Delta_i(A)$  depends only on  $H$  and  $i$ ; one defines  $\Delta_i(H) = \Delta_i(A)$ . Clearly  $\Delta_i(H) = 0$  for  $i \leq \text{rg } H = n - \text{rg } A$  and  $\Delta_i(H) \neq 0$  for  $i > \text{rg } H$ . The invariant  $\Delta_1(H)$  is denoted also by  $\text{ord } H$ ; it is called the order of  $H$ . It is clear that  $\text{ord } H \neq 0$  iff  $H = \text{Tors}_R H$ . For proofs and further information see [1].

Recall, finally, that a local ring is a domain  $K$  which has a unique maximal (proper) ideal. The quotient of  $K$  by this ideal is a field which we shall call "the field associated to  $K$ ".

3.2. LEMMA. *Let  $R, R'$  be (commutative) domains with 1 and let  $\varphi: R \rightarrow R'$  be a ring homomorphism. Let  $C = (\dots \rightarrow C_{i+1} \rightarrow C_i \rightarrow \dots)$  be a finitely generated free chain complex over  $R$  and let  $C'$  be the chain  $R'$ -complex  $R' \otimes_R C$ . Then: (i)  $\text{rk}_{R'} H_i(C') \geq \text{rk}_R H_i(C)$  and  $\text{rk } \partial'_i \leq \text{rk } \partial_i$  for all  $i$  where  $\partial_i, \partial'_i$  are the boundary homomorphisms  $C_{i+1} \rightarrow C_i, C'_{i+1} \rightarrow C'_i$ ; (ii) if  $\text{rk } H_i(C') = \text{rk } H_i(C)$  for some  $i$  then  $\text{rk } \partial'_j = \text{rk } \partial_j$  for  $j = i, i+1$ ; (iii) if  $R, R'$  are unique factorization Noetherian domains and if  $\text{rk } H_i(C') = \text{rk } H_i(C)$  then  $\varphi(\text{ord}(\text{Tors}_R H_i(C)))$  divides  $\text{ord}(\text{Tors}_{R'} H_i(C'))$ .*

*Proof.* Let  $n = \text{rk } C_i$ . Let  $A = (a_{p,q})$ ,  $1 \leq q \leq n$ ,  $1 \leq p$ , be the matrix of  $\partial_i$  with respect to some bases in  $C_i, C_{i+1}$ . Then  $A' = (\varphi(a_{p,q}))$  is the matrix of  $\partial'_i$  with respect to the induced bases in  $C'_i, C'_{i+1}$ . It is evident that  $\text{rk } \partial'_i = \text{rk } A' \leq \text{rk } A = \text{rk } \partial_i$ . Therefore

$$\text{rk } H_i(C') = n - \text{rk } \partial'_i - \text{rk } \partial'_{i+1} \geq n - \text{rk } \partial_i - \text{rk } \partial_{i+1} = \text{rk } H_i(C).$$

These inequalities imply (i) and (ii).

Put  $r = n - \text{rk } A + 1$  and denote the  $R$ -module  $C_i/\text{Im } \partial_i$  by  $J$ . Since  $A$  is a presentation matrix of  $J$  we have  $\text{ord}(\text{Tors}_R J) = \Delta_r(A)$  (see [1, p. 31]). From the exact sequence  $0 \rightarrow H_i(C) \rightarrow J \rightarrow C_{i-1}$  we obtain that  $\text{Tors } J = \text{Tors } H_i(C)$ . Thus  $\text{ord}(\text{Tors } H_i(C)) = \Delta_r(A)$ . Analogously  $\text{ord}(\text{Tors } H_i(C')) = \Delta_{r'}(A')$  where  $r' = n - \text{rk } A' + 1$ . If  $\text{rk } H_i(C) = \text{rk } H_i(C')$  then  $\text{rk } A = \text{rk } A'$  and therefore  $r = r'$ . It is evident that  $\varphi(\Delta_j(A))$  divides  $\Delta_j(A')$  for all  $j$ . This implies (iii).

3.3. LEMMA. Let  $R$  be a local ring and  $F$  be the associated field. Let  $f: C_1 \rightarrow C_0$  be a  $R$ -homomorphism of finitely generated free  $R$ -modules and let  $\bar{f}: F \otimes_R C_1 \rightarrow F \otimes_R C_0$  be the induced  $F$ -homomorphism. If  $\text{rk } f = \text{rk } \bar{f}$  then with respect to some bases in  $C_1, C_0$  the homomorphism  $f$  is presented by the matrix  $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$  where  $E$  is the unit matrix of order  $\text{rk } f$ .

*Proof.* Since  $F$  is a field we can choose bases  $d_0, d_1$  respectively in  $F \otimes_R C_0, F \otimes_R C_1$  so that the matrix of  $\bar{f}$  regarding these bases has the form  $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ . Let  $\mathcal{D}_i$  be a lifting of  $d_i$  to  $C_i, i = 1, 2$ . Here  $\mathcal{D}_i$  is a sequence of  $\text{rg } C_i$  elements of  $C_i$ . In view of Nakayama's lemma  $\mathcal{D}_i$  generate  $C_i$ . This implies that  $\mathcal{D}_i$  generates the  $(\text{rg } C_i)$ -dimensional vector space  $Q(R) \otimes_R C_i$  over the field  $Q(R)$ . Therefore, the elements of the sequence  $\mathcal{D}_i$  are linearly independent over  $Q(R)$  and, hence, over  $R$ . Thus  $\mathcal{D}_i$  is a basis of  $C_i$  for  $i = 0, 1$ . The matrix of  $f$  with respect to bases  $\mathcal{D}_0, \mathcal{D}_1$  has the form  $\begin{bmatrix} E+U & Z \\ X & Y \end{bmatrix}$  where  $U, X, Y, Z$  are matrices over the maximal ideal  $u$  of  $R$ . Note that  $\det(E+U) = 1 \pmod{u}$ . Since all elements of  $R \setminus u$  are invertible in  $R$  the square matrix  $E+U$  is invertible over  $R$ . Therefore we can choose bases in  $C_0, C_1$  so that the corresponding matrix of  $f$  equals  $\begin{bmatrix} E & 0 \\ 0 & Y' \end{bmatrix}$ . Since  $\text{rk } f = \text{rk } \bar{f} = \text{rk } E, Y' = 0$ .

3.4. LEMMA. Let  $R$  be a local ring and  $F$  be the associated field. Let  $C = (\dots \rightarrow C_{i+1} \rightarrow C_i \rightarrow \dots)$  be a finitely generated free chain complex over  $R$ . Let  $C'$  be the chain  $F$ -complex  $F \otimes_R C$ . Let  $\partial_i, \partial'_i$  be the boundary homomorphisms  $C_{i+1} \rightarrow C_i, C'_{i+1} \rightarrow C'_i$ . If  $\text{rk}_R H_i(C) = \text{rk}_F H_i(C')$  for some  $i$  then:  $H_i(C), \text{Im } \partial_{i+1}, \text{Im } \partial_i$  are free  $R$ -modules and  $C_i = \text{Im } \partial_{i+1} \oplus H_i(C) \oplus \text{Im } \partial_i$ ; the projection  $C \rightarrow C'$  induces  $F$ -isomorphisms  $F \otimes_R H_i(C) \rightarrow H_i(C'), F \otimes_R \text{Im } \partial_j \rightarrow \text{Im } \partial'_j$  with  $j = i, i+1$ .

This Lemma directly follows from Lemmas 3.2 (ii) and 3.3.

#### § 4. PROOF OF THEOREMS 1 AND 2

4.1. PROOF OF THEOREM 1. Denote by  $Q_n$  the fraction field of the ring  $\Lambda_n = \mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ . Denote by  $Q_n^0$  the subring of  $Q_n$  which consists of rational functions  $fg^{-1}$  with  $f, g \in \Lambda_n$  and  $g \notin (t_n - 1)\Lambda_n$  (so that