

2. A Topological Lemma

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **08.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

from $\{u \in C^\infty(X), \int u dX_g = 0\}$ to $\{v \in C^\infty(X), \int v dX_{g'} = 0\}$ ($dX_{g'}$ denotes the volume form in the metric g') when $\lambda = 0$.

For completeness, let us indicate how, for instance theorem 0.2, can be reduced to equation (1) with $\lambda = 0$. It is quite straightforward. First of all we are given a cohomology class $c \in H^2(X, \mathbf{R})$ such that there exists a Kähler form ω in c ; let ρ be the Ricci form of ω : $\rho \in C_1(X)$, the first Chern class of X .

Then we are given $\gamma \in C_1(X)$ and hence $f \in C^\infty(X)$ a real function (defined up to an additive constant), which measures the deviation for ω from satisfying 0.2:

$$\gamma - \rho = \sqrt{-1} \partial \bar{\partial} f.$$

Now we look for another Kähler form $\omega' \in c$, i.e. we look for a smooth real function φ (also defined up to an additive constant), where

$$\omega' - \omega = \sqrt{-1} \partial \bar{\partial} \varphi$$

such that the Ricci form ρ' of ω' coincides with γ .

In other words, we want φ to satisfy

$$\rho' - \rho \equiv \sqrt{-1} \partial \bar{\partial} f,$$

or equivalently, if g and g' are the Kähler metrics respectively associated with ω and ω' ,

$$\partial \bar{\partial} \{-\text{Log det}(g'g^{-1})\} \equiv \partial \bar{\partial} f$$

which immediately yields equation (1) with $\lambda = 0$:

$$-\text{Log det}(g'g^{-1}) = f,$$

since anyway f is only defined up to an additive constant.

As ω and ω' are cohomologous and closed, so are the corresponding volume forms, therefore X has same volume measured with the metrics g and g' ; this defines completely f , subject to the natural constraint mentioned above.

2. A TOPOLOGICAL LEMMA

In our setting, the continuity method becomes a "surjectivity method" since it is based on the following

LEMMA 2.1. Let A, B be metric spaces, with $A \neq \emptyset$ and B connected. Let $P: A \rightarrow B$ be a continuous map. Assume:

- (i) P is open,
- (ii) P is proper, that is, for any compact subset K in B , $P^{-1}(K)$ is compact. Then P is surjective.

Proof. We only need to prove that $P(A)$ is closed. Let b_0 be a point in $\overline{P(A)}$. Since B is a metric space, there exists a sequence $(b_i)_{i>0}$ in $P(A)$ converging to b_0 . The subset $K = \{b_0, b_1, b_2, \dots\}$ is compact, hence so is $PP^{-1}(K)$. The latter contains $b_1, \dots, b_i, \dots$, hence b_0 , and it is obviously contained in $P(A)$. Q.E.D.

In order to make use of this lemma, we shall need some inverse function theorem for (i), and some *a priori* estimates for (ii).

3. LOCAL INVERSION

THEOREM 3.1. Let X be a smooth compact manifold, V and W smooth vector bundles on X , U an open set in $C^\infty(X, V)$, and $P: U \rightarrow C^\infty(X, W)$, a smooth nonlinear elliptic partial differential operator. Let A and B be LCFC submanifolds of U and of $C^\infty(X, W)$ respectively, such that the restriction P_A of P to A , sends A into B . Then the Jacobian criterion holds for P_A , namely, if the derivative of $P_A: A \rightarrow B$ is invertible at $\varphi_0 \in A$, then P_A is a local diffeomorphism near φ_0 .

This is a convenient variant of the Nash-Moser theorem (e.g. [14]) regarding suitable restrictions of elliptic operators. It is established in a separate paper [11] (see also [22]). It relies only on the classical (Banach) inverse function theorem combined with *elliptic regularity*.

Remark 3.2. The Nash-Moser theorem has been studied by many authors, see the bibliography below and further references in [14] [15] [25].

4. PROPERNESS

In view of (2), theorem 3.1 implies that P_λ is open. We want to apply lemma 2.1 in order to prove that P_λ is surjective from A_λ to B_λ . Since $P_\lambda(A_\lambda) \neq \emptyset$ (it contains 0), and since B_λ is connected, this amounts to proving that P_λ is *proper*. Let us explain why *a priori* estimates imply properness.

Concerning subsets in A_λ we have