

# 0. Introduction

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## ISOCLINIC $n$ -PLANES IN $R^{2n}$ AND THE HOPF-STEENROD SPHERE BUNDLES $S^{2n-1} \rightarrow S^n$ , $n = 2, 4, 8$

by Yung-Chow WONG and Kam-Ping MOK

### 0. INTRODUCTION

The construction of the sphere bundles  $S^{2n-1} \rightarrow S^n$ ,  $n = 2, 4, 8$ , by N. Steenrod was accomplished in an ingenious but rather roundabout way, using the famous Hopf maps and the systems of complex numbers, quaternions and Cayley numbers (cf. Hopf [2], Steenrod [5, pp. 105-110] and Hilton [1, pp. 51-55]). In this paper, we show how the theory of mutually isoclinic  $n$ -planes in a real Euclidean  $2n$ -space  $R^{2n}$  as developed by Wong in [8, 9] enables us to reconstruct these sphere bundles in a more natural manner by working strictly within the field of real numbers and giving the three cases  $n = 2, 4, 8$  a more unified treatment. In addition, we prove that the bundle group  $O(8)$  of the Hopf-Steenrod sphere bundle  $S^{15} \rightarrow S^8$  can be replaced by  $SO(8)$  but not by any subgroup of  $SO(8)$ .

In § 1, we recall certain results on maximal sets of mutually isoclinic  $n$ -planes in  $R^{2n}$  that motivated our investigation. In § 2, we confine ourselves to the cases  $n = 2, 4, 8$ , and prove some results that will be used later. In § 3, we construct three sphere bundles by using maximal sets of mutually isoclinic  $n$ -planes in  $R^{2n}$ . In § 4, we give a unified and explicit formulation of the three Hopf-Steenrod sphere bundles, using as Steenrod did the Hopf maps and systems of complex numbers, quaternions and Cayley numbers. In § 5, we prove that the Hopf maps and maximal sets of mutually isoclinic  $n$ -planes in  $R^{2n}$ ,  $n = 2, 4, 8$ , are equivalent concepts, and that the reformulated Hopf-Steenrod sphere bundles described in § 4 are topologically essentially the same as the sphere bundles constructed in § 3. The paper ends with two appendices in which we explain the operations of Cayley numbers, and give a direct proof that for  $n = 2, 4$ , or  $8$ , the  $n$ -planes in  $R^{2n}$  containing the Hopf fibers of  $S^{2n-1}$  are mutually isoclinic  $n$ -planes.

In a continuation of this paper being prepared, we shall show that the image of the Hopf fibers of  $S^{2n-1}$ ,  $n = 2, 4$ , or  $8$ , under an inversion in  $R^{2n}$  has some very interesting properties which include those recently found by J. B. Wilker [7] for the case  $n = 2$ .

We wish to thank Prof. Wilker for letting us have a preprint of his paper, and Prof. Kee-Yuen Lam for some helpful discussions.

# 1. SOME RESULTS ON ISOCLINIC $n$ -PLANES IN $R^{2n}$

By a Euclidean (vector)  $m$ -space  $R^m$ , where  $m$  is a positive integer, we mean an  $m$ -dimensional vector space provided with a positive definite inner product. An  $r$ -plane ( $1 \leq r \leq m-1$ ) in  $R^m$  is an  $r$ -dimensional vector subspace of  $R^m$  provided with the induced inner product. In  $R^m$ , length of a vector, angle between two vectors, orthogonality between a  $k$ -plane and an  $r$ -plane, (orthogonal) projection of a vector on an  $r$ -plane, orthonormal bases and rectangular coordinates are defined in the usual way.

In an  $R^{2n}$ , let  $\mathbf{A}$ ,  $\mathbf{B}$  be any two  $n$ -planes. Then we say that  $\mathbf{A}$  is *isoclinic* with  $\mathbf{B}$  at angle  $\theta$  if the angle between every nonzero vector in  $\mathbf{A}$  and its projection on  $\mathbf{B}$  is always equal to  $\theta$ . It turns out that if  $\mathbf{A}$  is isoclinic with  $\mathbf{B}$  at angle  $\theta$ , then  $\mathbf{B}$  is isoclinic with  $\mathbf{A}$  at the same angle  $\theta$ . Therefore, in this case, we shall say that  $\mathbf{A}$  and  $\mathbf{B}$  are isoclinic at angle  $\theta$ , or simply,  $\mathbf{A}$  and  $\mathbf{B}$  are isoclinic.

A set  $\Phi$  of  $n$ -planes in  $R^{2n}$  is said to be a *maximal set of mutually isoclinic  $n$ -planes* if every pair of  $n$ -planes in  $\Phi$  are isoclinic and  $\Phi$  is not contained in a larger set of mutually isoclinic  $n$ -planes. It is easy to see from definition that if  $\mathbf{A}$  is isoclinic with  $\mathbf{B}$  at angle  $\theta$ , then its orthogonal complement  $\mathbf{A}^\perp$  is isoclinic with  $\mathbf{B}$  at angle  $\frac{\pi}{2} - \theta$ . Consequently, if  $\Phi$  is any maximal set of mutually isoclinic  $n$ -planes in  $R^{2n}$  and  $\mathbf{A} \in \Phi$ , then  $\mathbf{A}^\perp \in \Phi$ .

In his memoir [8] Wong determined, for each  $n$ , the dimensions of the maximal sets of mutually isoclinic  $n$ -planes in  $R^{2n}$ , the number of non-congruent maximal sets of a given dimension, and explicit equations of the  $n$ -planes in any maximal set of mutually isoclinic  $n$ -planes containing a given  $n$ -plane.

In the following, we summarize some of his results related to the problem studied in this paper.

**THEOREM 1.1.** (Wong [8, pp. 25-26]). *In  $R^{2n}$  provided with a rectangular coordinate system  $(x, y) \equiv ([x_1 \dots x_n], [x_{n+1} \dots x_{2n}])$ , any maximal set  $\Phi$  of mutually isoclinic  $n$ -planes containing the  $n$ -plane  $\mathbf{O}: y = 0$  (and consequently, also the  $n$ -plane  $\mathbf{O}^\perp: x = 0$ ) is congruent to the set of  $n$ -planes with equations*