1. The Monge-Ampère equation

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note is to analyze how far these tools are necessary for the proof. It turns out that it is possible to reduce the contribution of elliptic theory mainly to a suitable local inverse function theorem for nonlinear elliptic operators acting on smooth functions [22] [11].

The proof presented below deals only with the reduction to the crucial estimates of order zero, two and three, already obtained by Aubin and Yau. Although it is not so clear in [21] [24] these estimates were performed essentially through coordinate free tensor calculus. We show how higher order estimates may be obtained in the same way.

The whole approach applies as well to the corresponding *real* elliptic Monge-Ampère equation on compact Riemannian manifolds [9] and to various generalizations of it. We shall freely use arguments of Calabi [6, 7, 8], Aubin [1, 2, 3], Yau [23, 24], Bourguignon *et al.* [5] [21].

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1. THE MONGE-AMPÈRE EQUATION

Let X be a compact connected finite-dimensional Kähler manifold. ω denotes the original C^∞ Kähler form, g the corresponding Kähler metric, $\varphi \in C^\infty(X)$ denotes a C^∞ real-valued function on X, and we set

$$\omega' = \omega + \sqrt{-1}\, \partial\bar{\partial}\varphi$$

where ∂ and $\bar{\partial}$ are the usual first order differential operators. Let g' denotes the Kähler metric corresponding to ω'. In the sequel, "smooth" means C^∞.

If g and g' are viewed as morphisms from the antiholomorphic tangent bundle into the holomorphic cotangent bundle T^*, then $(g'g^{-1})$ is an endomorphism of T^* the determinant of which, $\det(g'g^{-1})$ is a smooth function on X. The function φ is said to be *admissible* if and only if $\det(g'g^{-1})$ is strictly positive on X. One proves easily that if φ is admissible, then g' is again a (positive definite Kähler) metric e.g. [2], p. 119.

Let $\lambda \in [0, +\infty)$. It is convenient to denote by A_λ the subset of $C^\infty(X)$ consisting in all admissible real-valued smooth functions φ on X, satisfying, in case $\lambda = 0$ the further zero average condition

$$\int_X \varphi dX_g = 0,$$

where dX_g denotes the volume form associated with g. When $\lambda > 0$, A_λ is an *open* subset of $C^\infty(X)$: indeed, the natural injection $C^\infty(X) \hookrightarrow C^2(X)$ is continuous, with respect to the Fréchet and Banach topologies; A_λ is the pull back, by this injection of the *open* set

$$\{\varphi \in C^2(X),\ \det(g'g^{-1}) > 0\}.$$

Definition 1.1. Let X be a smooth compact manifold, V a smooth vector bundle on X, $C^\infty(X, V)$ the Fréchet space of smooth sections of V. A *LCFC submanifold* of $C^\infty(X, V)$, is a locally closed finite codimensional Fréchet submanifold of $C^\infty(X, V)$.

The set A_0 is an *open* subset of the LCFC submanifold

$$\{\varphi \in C^\infty(X),\ \int_X \varphi dX_g = 0\}.$$

We define the map P_λ, from A_λ to $C^\infty(X)$, by

$$P_\lambda(\varphi) = \lambda\varphi - \operatorname{Log}\det(g'g^{-1}).$$

The proofs of theorems 0.1 and 0.2 have been reduced to the solution, when $\lambda \geqslant 0$, of the following complex Monge-Ampère equation (e.g. [21], (lecture n° V), [4] p. 143):

$$P_\lambda(\varphi) = f, \tag{1}$$

where $f \in C^\infty(X)$ is *given*, and in case λ vanishes, has to satisfy the natural constraint (e.g. [1] p. 403, [24] p. 361, [21] p. 85),

$$\int_X e^{-f} dX_g = \int_X dX_g.$$

In any case, f ranges in a connected LCFC submanifold B_λ of $C^\infty(X)$. To see that B_0 is *connected*, notice that $0 \in B_0$ and that given any $f \in B_0$, the following path connects f to 0 in B_0:

$$t \in [0, 1] \to f_t =: tf + \operatorname{Log}(\int_X e^{-tf} dX_g) - \operatorname{Log}(\int_X dX_g).$$

The derivative of the map P_λ at $\varphi \in A_\lambda$, is given by

$$dP_\lambda(\varphi, \delta\varphi) = (\Delta' + \lambda)\,\delta\varphi \tag{2}$$

where Δ' stands for the Laplace operator on functions in the metric g' [21] p. 96. Classically, it follows from the Maximum Principle [20], the Fredholm Alternative theorem and from the elliptic regularity theory, that $dP_\lambda(\varphi, \cdot)$ is *invertible* $\forall \varphi \in A_\lambda$, either from $C^\infty(X)$ to itself when $\lambda > 0$, or from $\{u \in C^\infty(X), \int u dX_g = 0\}$ to $\{v \in C^\infty(X), \int v dX_{g'} = 0\}$ ($dX_{g'}$ denotes the volume form in the metric g') when $\lambda = 0$.

For completeness, let us indicate how, for instance theorem 0.2, can be reduced to equation (1) with $\lambda = 0$. It is quite straightforward. First of all we are given a cohomology class $c \in H^2(X, \mathbf{R})$ such that there exists a Kähler form ω in c; let ρ be the Ricci form of ω: $\rho \in C_1(X)$, the first Chern class of X.

Then we are *given* $\gamma \in C_1(X)$ and hence $f \in C^\infty(X)$ a real function (defined up to an additive constant), which measures the deviation for ω from satisfying 0.2:

$$\gamma - \rho = \sqrt{-1}\, \partial\bar{\partial} f .$$

Now we look for another Kähler form $\omega' \in c$, i.e. we look for a smooth real function φ (also defined up to an additive constant), where

$$\omega' - \omega = \sqrt{-1}\, \partial\bar{\partial}\varphi$$

such that the Ricci form ρ' of ω' *coincides with* γ.

In other words, we want φ to satisfy

$$\rho' - \rho \equiv \sqrt{-1}\, \partial\bar{\partial} f ,$$

or equivalently, if g and g' are the Kähler metrics respectively associated with ω and ω',

$$\partial\bar{\partial}\{-\operatorname{Log}\det(g'g^{-1})\} \equiv \partial\bar{\partial} f$$

which immediately yields equation (1) with $\lambda = 0$:

$$-\operatorname{Log}\det(g'g^{-1}) = f ,$$

since anyway f is only defined up to an additive constant.

As ω and ω' are cohomologous and closed, so are the corresponding volume forms, therefore X has same volume measured with the metrics g and g'; this defines completely f, subject to the natural constraint mentioned above.

2. A Topological Lemma

In our setting, the continuity method becomes a "surjectivity method" since it is based on the following