§5. Monopoles and Instantons

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Remarks. 1) It would be interesting to see what kind of harmonic representatives for classes in $H^1(M; \mathbf{R})$ can be found.

2) Theorem 4.2 generalizes to identify elements of $H^j(M, \delta M; \mathbf{R})$ with L^2 harmonic forms for any oriented n-dimensional Riemannian manifold M for which a conformal compactification of $M \times S^k$ exists, for all k, provided $j < n/2$.

§ 5. Monopoles and Instantons

Our goal is now to exploit the compactification X of $M \times S^1$ (see § 2) to get monopoles on M from S^1-invariant instantons on X. We shall also relate the instanton number on X to various topological invariants of the monopoles on M. General background for this section can be found in Freed-Uhlenbeck [12] and Jaffe-Taubes [22]. More specifically our approach here is very similar to the one taken in Atiyah [2].

Let P be a principal $SU(2)$-bundle over X, with $c_2(P) = k \geqslant 0$. Recall that X comes naturally with a conformal structure. This enables us to talk about *instantons or anti-self-dual connections* A on P. These are defined to be the solutions of the *anti-self-duality equation*:

5.1 $$F^A = - *_4 F^A \quad (*_4 \text{ the Hodge star on } \Lambda^2(X)) .$$

Here F^A is the curvature of A, a section of $\Lambda^2(X) \otimes g_P$ with $g_P = P \times_{Ad} su(2)$. The instantons are the absolute minima of the *Yang-Mills functional*:

5.2 $$YM(A) = (16\pi^2)^{-1} \int_X < F^A \wedge *F^A >$$

where $<\alpha, \beta> = - 2 \cdot tr(\alpha\beta)$ is an invariant inner product on $su(2)$. For an instanton $YM(A) = k$.

Next assume that the double cover $\tilde{S}^1$ of S^1 acts on P by bundle automorphisms, covering the action on X; the double cover will be needed in order to include the spin bundles of X. Our interest will now lie in $\tilde{S}$-invariant instantons on P. To relate these to objects on M introduce the map:

$$j : M \to X : m \to i'(m, 1) \quad (\text{compare } 2.2) ,$$

which is a diffeomorphism onto its image. Let v be the vectorfield on P induced by the $\tilde{S}^1$-action. If we interprete an $\tilde{S}^1$-invariant connection A as a 1-form on P, then define the Higgs-field Φ to be the $su(2)$-valued function $j^*A(\frac{1}{2}v)$ on j^*P. It is easy to see that Φ is a section of j^*g_P.

Further $A_3 = j^*A$ defines a connection on the bundle j^*P over M. A little computation shows that the $\tilde{S}^1$-invariant connection A is anti-self-dual iff (A_3, Φ) satisfy the so called *Bogomol'nyi equation* on M:

5.3 $$d^{A_3}\Phi = - *_3F^{A_3}.$$

As 5.3 is the standard equation describing *magnetic monopoles* on three dimensional manifolds, this leads to the definition.

Definition 5.1. A monopole on P is an $\tilde{S}^1$-invariant instanton on P.

Normally one defines a monopole by imposing certain asymptotic conditions rather than requiring it to extend over a compact manifold. In Braam [10] it is explained that results of the Sibners imply that this amounts to the same. We shall see below that the boundary data are the same.

If $GA(P)$ denotes the group of $\tilde{S}$-invariant gauge transformations on P, then $GA(P)$ leaves the set of monopoles invariant. Just as for instantons one can therefore define a *monopole moduli space*, equal to:

5.4 $$\{\text{solutions of } 5.3\}/GA(P)$$

In Braam [10] is shown that under some assumptions these moduli spaces are non-empty finite dimensional manifolds.

We shall now return to our $\tilde{S}^1$-equivariant bundle P and relate topological invariants of the action to asymptotic invariants of (A_3, Φ) on M. Restricted to one of the fixed surfaces S_j, $\tilde{S}^1$ acts by gauge transformations on P. The fibres of $E = P \times_{SU(2)} \mathbf{C}^2$ over S_j decompose into eigenspaces for the $\tilde{S}^1$ action. Denote by $m_j \in \mathbf{Z}_{\geqslant 0}$ the $\tilde{S}^1$-weight which is non-negative.

If $m_j > 0$ then:

5.5 $$E_{|S_j} \cong L_j \oplus L_j^*$$

where L_j is the complex line bundle in E of weight m_j and L_j^* that of weight $- m_j$; because $c_1(E_{|S_j}) = 0$, L_j^* is also the dual of L_j. In order to define the first Chern classes of L_j it is convenient to have an orientation of S_j. Recall that X is oriented and that a neighbourhood of S_j in X looks like $S_j \times \mathbf{R}^2$. The $\mathbf{R}^2$ is oriented by the S^1-action, and this induces an orientation of S_j. Now write $c_1(L_j) = - k_j \cdot x_j$ with $k_j \in \mathbf{Z}$ and x_j the positive generator of $H^2(S_j; \mathbf{Z})$. If $m_j = 0$ then $E_{|S_j}$ is trivial as an $\tilde{S}^1$-equivariant vector bundle. We shall leave k_j undefined in this case.

There is one important constraint on the m_j. This becomes clear by remarking that $- 1 \in \tilde{S}^1$ acts as a gauge transformation on all of E, i.e. as

$+1$ or as -1. This implies that either all m_j are even or they are all odd. In Braam [10] we have shown that any set of invariants (m_j, k_j) satisfying this constraint arises from a suitable $\tilde{S}^1$-equivariant bundle, and that the $\tilde{S}^1$-isomorphism class is determined by (m_j, k_j).

Definition 5.2. The moduli space of monopoles on a principal $SU(2)$-bundle P with invariants (m_j, k_j) will be denoted by $\mathcal{M}(m_j, k_j)$.

Having defined the relevant invariants of P, the question now arises what they amount to in terms of asymptotic conditions for a pair (A_3, Φ) on M. The vector field v on P turns vertical over S_j. This shows that:

5.6 $$|\Phi(y)| \to m_j \quad \text{if} \quad y \to S_j \subset \delta M .$$

This is the Prasad-Sommerfeld boundary condition used in physics and the numbers m_j are called the *masses* of the monopole.

The solutions of the Bogomol'nyi equation 5.3 are minima of the *energy functional*:

5.7 $$E(A_3, \Phi) = (8\pi)^{-1} \int_M |F^{A_3}|^2 + |d_{A_3}\Phi|^2 \, dV_3 .$$

If the pair (A_3, Φ) arises from an invariant connection A on P then $E(A_3, \Phi) = YM(A)$. If we assume that (A_3, Φ) satisfies 5.4, then:

$$|d_{A_3}\Phi|^2 \, dV_3 = |F^{A_3}|^2 \, dV_3 = \langle F^{A_3} \wedge d_{A_3}\Phi \rangle = d\langle F^{A_3} \cdot \Phi \rangle ,$$

by the Bianchi identity. It follows that:

$$E(A_3, \Phi) = -2 \textstyle\sum_j (8\pi)^{-1} \cdot \int_{S_j} \langle F^{A_3} \cdot \Phi \rangle .$$

The minus sign appears because the boundary orientation of S_j does not agree with orientation we have given it above. A moments reflection shows that $2 \cdot (8\pi)^{-1} \cdot \int_{S_j} \langle F^{A_3} \cdot \Phi \rangle = -m_j \cdot k_j$. Putting things together we get:

5.7 $$\textstyle\sum m_j \cdot k_j = E(A_3, \Phi) = YM(A) = k .$$

This is essentially the localization formula in equivariant cohomology applied to the equivariant $c_2(P)$, see Atiyah [2].

Exactly what the physical symmetry breaking would lead one to expect does indeed happen: far away in M, that is near an S_j with $m_j \neq 0$, the connection almost becomes a $U(1)$-connection on L_j, the bundle of eigenvectors of Φ of eigenvalue $\frac{1}{2} \cdot m_j$. The *charges* k_j appear as first Chern classes of these line bundles on the boundary surfaces. This is of course nothing but the quantized charge of a $U(1)$-monopole, a so called Dirac monopole, on L_j. Dirac monopoles have singularities, but the genuine non-

Abelian character of $SU(2)$-monopoles in the core of M allows for non-singular solutions.

From 5.7 we see that $\sum m_j \cdot k_j \geqslant 0$ is necessary for the existence of monopoles, however this is by no means sufficient as we shall see below (also compare Braam [10]).

We shall end this section by giving some simple examples of monopoles.

Examples 5.3. 1) Monopoles with all $m_j = 0$. For these monopoles $YM(A) = 0$, so we are dealing with flat connections. The Higgs field Φ vanishes, this follows from the Bogomol'nyi equation. It is not hard to see that the moduli space $\mathscr{M}(0, 0)$ equals the space of all representations $\pi_1(X) \to SU(2)$ modulo conjugacy: one assign to a flat connection its holonomy representation. This space can be very non-trivial; e.g. if $M = H^3/$ Fuchsian group $\cong S \times \mathbf{R}$, with S a surface, then $\mathscr{M}(0, 0)$ is the space of representations of $\pi_1(S) \to SU(2)$ modulo conjugacy. By the theorem of Narasimham-Seshadri this is the same as the moduli space of semi-stable $SL(2, \mathbf{C})$-bundles on S, for any complex structure on S. The topology of this $\mathscr{M}(0, 0)$ was investigated by Atiyah-Bott [4].

2) Next keep $k_j = 0$ but take at least one m_j to be nonzero. The connections are still flat so Φ is covariantly constant. This shows that $\mathscr{M}(m_j, 0) = \emptyset$ unless all m_j are equal. Further

$$\begin{aligned}\mathscr{M}(m, o) &\cong \operatorname{Repr}\left(\pi_1(M), S^1\right) \cong \operatorname{Repr}\left(H_1(M; \mathbf{Z}), S^1\right) \\ &\cong H_1(X; \mathbf{Z})_{tor} \times \{H_1(X; \mathbf{R})/H_1(X; \mathbf{Z})\}.\end{aligned}$$

3) For $M \cong H^3$ all monopoles were determined by Atiyah [2]. The moduli space $\mathscr{M}(m, k)$ equals $\{\phi: S^2 \to S^2;\ \phi$ rational, degree $\phi = k$, $\phi(\infty) = 0\}$, modulo multiplication by complex scalars of length 1. The monopole associated to the rational function $\sum_j \exp(i\alpha_j) \cdot \lambda_j/(z - a_j)$ with $\lambda_j \in \mathbf{R}_{>0}$, $a_j \in \mathbf{C}$, represents k lumps, centered at approximately $(a_j, \lambda_j) \in \mathbf{R}^3_+ \cong H^3$, with relative phase factors $\exp\left(i(\alpha_{j_1} - \alpha_{j_2})\right)$.

4) Monopoles arising from Riemannian curvature. If X is a oriented Riemannian 4-manifold then one can write the curvature tensor $R: \Lambda^2 \to \Lambda^2$ as $\begin{bmatrix} W_+ + (R_{sc}/3) & B \\ B^* & W_- + (R_{sc}/3) \end{bmatrix}$ relative to the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, in which B equals the Ricci curvature and $W_\pm$ the Weyl tensor. If X is a conformally flat spin manifold with a metric of zero scalar curvature then this curvature tensor equals $\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$. It follows that the connection

on the spin bundle S_+ is anti-self-dual. Recall (see § 3) that for Γ Fuchsian, extended Fuchsian or a suitable Schottky group X_Γ admits such a metric. The connection on S_+ is a monopole because the metrics are S^1-invariant. The mass(es) is (are) 1 by proposition 2.2, and the charges k_j equal $g - 1$, where g is the genus of the fixed surface(s). Choosing a different spin structure amounts to tensoring the bundle with a 2-torsion element in $\mathrm{Repr}\,(\pi_1(M), S^1)$, compare 2).

In section 7 we shall come to grips with explicit formulae for nontrivial monopoles on certain handlebodies. In Braam-Hurtubise [11] the moduli spaces of monopoles on a solid torus are investigated in considerable detail. A general existence theory for monopoles on hyperbolic manifolds has been developed in Braam [10].

§ 6. Twistor spaces

To a conformally flat oriented 4-manifold X there are naturally associated two complex manifolds Z_+ and Z_-, the *twistor spaces* of X. Applying our construction of § 2 we thus get twistor spaces for hyperbolic 3-manifolds. It will be shown here that these carry a lot of geometric information associated to the 3-manifold M, such as the complete geodesic flow. Also they allow for a description of monopoles through holomorphic geometry. For the rest of this section let X be the conformal compactification of $M \times S^1$, with M a hyperbolic 3-manifold H^3/Γ as in § 2. We shall state those properties of $Z_\pm$ that we will need, and refer to Atiyah [1] and Atiyah-Hitchin-Singer [5] for proofs and more details. The general line of thought in this section is very similar to that of Hitchin [20] and Atiyah [2].

If $S_+(S_-)$ is the spin bundle of positive (negative) chirality on X, then $Z_+(Z_-)$ can be realised as the $\mathbf{CP}^1$-bundles over X:

$$P(S_+) \to X \qquad \big(P(S_-) \to X\big),$$

where $P(\ )$ denotes projectivization of vectorbundles. A remarkable fact is that Z_+ and Z_- are *complex manifolds* with a complex structure encoded in the conformal structure of X. However, the twistor spaces are only Kähler if $X \cong S^4$ or $X \cong \mathbf{CP}^2$, which in our case results in $\Gamma = \{e\}$ (see Hitchin [19]). There is an orientation reversing isometry of X arising from conjugation of the circles. This interchanges the two spin bundles and makes