

Appendix 1. The Cayley numbers

Objektyp: **Appendix**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

correspond to the coordinate transformations $t \rightarrow tB(\lambda)/N(\lambda)^{1/2}$ in $\mathcal{I}_4$, where $B(\lambda)$ are the matrices given in (1.7) in Theorem 1.6. By Theorem 2.5, the elements $B(\lambda)/N(\lambda)^{1/2}$ of $SO(4)$ form a subgroup isomorphic with S^3 . Therefore, the bundle group $O(4)$ in $\mathcal{H}\mathcal{S}_4$ can be replaced by S^3 . Similarly, the bundle group $O(2)$ in $\mathcal{H}\mathcal{S}_2$ can be replaced by S^1 . With these observations, we can now prove the following theorem by proceeding as in the proof of Theorem 5.3.

THEOREM 5.4. *The representative coordinate bundles constructed in § 4 for the sphere bundles $\mathcal{H}\mathcal{S}_2$ and $\mathcal{H}\mathcal{S}_4$, with bundle groups S^1 and S^3 respectively, are topologically the same as the representative coordinate bundles constructed in § 3 for the sphere bundles $\mathcal{I}_2$ and $\mathcal{I}_4$, respectively.*

Finally, we remark that representative coordinate bundles of the bundles $\mathcal{S}\mathcal{L}_n$ in Theorem 4.2 are topologically essentially the same as the representative coordinate bundles of the bundles $\mathcal{I}\mathcal{L}_n$ in Theorem 3.2.

APPENDIX 1. THE CAYLEY NUMBERS

The Cayley numbers, denoted by X, Y, Z, W , etc. are ordered pairs (q_1, q_2) of quaternions subject to the rules and having the properties listed below. The set of all Cayley numbers, therefore, forms a (non-commutative and non-associative) real division algebra. No proof of the properties will be given as they can all be checked by direct computations.

(1) The *addition* is defined by

$$(q_1, q_2) + (q'_1, q'_2) = (q_1 + q'_1, q_2 + q'_2).$$

The *zero* is $O = (O, O)$.

(2) The *multiplication* is defined by

$$(q_1, q_2)(q'_1, q'_2) = (q_1q'_1 - q'_2{}^*q_2, q'_2q_1 + q_2q'_1{}^*),$$

where $q'_1{}^*, q'_2{}^*$ are respectively the conjugates of (the quaternions) q'_1, q'_2 . The (two-sided) *unit* is $1 \equiv (1, 0)$.

(3) Multiplication is

(i) distributive with respect to addition, i.e.,

$$(X + Y)W = XW + YW, \quad W(X + Y) = WX + WY;$$

- (ii) not commutative, i.e., generally, $XY \neq YX$ (but see (4) (iv) below);
 - (iii) not associative, i.e., generally, $(XY)W \neq X(YW)$ (but see (7) below).
- (4) The *real part* of $X \equiv (q_1, q_2)$ is $\text{Re } X = (\text{Re } q_1, 0) \equiv \text{Re } q_1$. X is said to be *real* if $X = \text{Re } X$; i.e., (q_1, q_2) is real iff q_1 is real and $q_2 = 0$.
- (i) $\text{Re}(X + Y) = \text{Re}(X) + \text{Re}(Y)$.
 - (ii) $\text{Re}(XY) = \text{Re}(YX)$.
 - (iii) $\text{Re}(CX) = 0$ for all X implies that $C = 0$.
 - (iv) $CX = XC$ for all X iff C is real. In this case, $C = (c_1, 0)$, where $c_1 = \text{real}$, and $CX = (c_1q_1, c_1q_2) = XC$.
- (5) The *conjugate* of $X \equiv (q_1, q_2)$ is $X^* = (q_1^*, -q_2)$.
- (i) $(X + Y)^* = X^* + Y^*$,
 - (ii) $(XY)^* = Y^*X^*$.
 - (iii) $X^* = X$ iff X is real.
- (6) The *norm* of X is the non-negative real number $N(X) \equiv XX^*$, which is also equal to X^*X . The *length* of X is the non-negative real number $|X| \equiv N(X)^{1/2} = (XX^*)^{1/2}$.
- (i) $N(X) = 0$ iff $X = 0$.
 - (ii) If $X \neq 0$, then $X^{-1} \equiv X^*/N(X)$ is a right and left inverse of X .
 - (iii) $N(XY) = N(X)N(Y)$. It follows from this that $XY = 0$ iff $X = 0$ or $Y = 0$.
- (7) Though multiplication is generally non-associative,
- (i) $(XY)Y^* = X(Y Y^*)$.
 - (ii) If $Y \neq 0$, then $(XY)Y^{-1} = X = Y^{-1}(YX)$.
 - (iii) $\text{Re}((XY)W) = \text{Re}(X(YW))$.

APPENDIX 2. THE HOPF FIBERING AND MUTUALLY ISOCLINIC PLANES

At the beginning of § 4, we described how H. Hopf obtained his fibering of S^{2n-1} by S^{n-1} over S^n , $n = 2, 4$, or 8 , by intersecting the unit sphere S^{2n-1} in $R^{2n} = Q_n \times Q_n$ with the Q_n -lines $Y = CX$ and $X = 0$. In Theorem 5.2, we proved that the Hopf fibering and maximal set of mutually isoclinic n -planes in R^{2n} are equivalent concepts. Here we prove, directly, the