

# 1. Example of a Stein space $X$ with $\widetilde{O(X)} \neq O(\widetilde{X})$

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

An analytic consequence of the construction presented here is that the normalization  $\tilde{X}$  of an irreducible Stein space  $X$  is  $\widetilde{\mathcal{O}(X)}$ -convex,  $\widetilde{\mathcal{O}(X)}$ -separable and has local coordinates by functions in  $\widetilde{\mathcal{O}(X)}$ . Some algebraic results are that  $\mathcal{O}(\tilde{X})$  is completely normal and that the two algebras  $\widetilde{\mathcal{O}(X)}$  and  $\mathcal{O}(\tilde{X})$  are always locally equal, i.e. their localizations at all maximal ideals in  $\mathcal{O}(X)$  are equal.

In this paper, a complex space refers to a reduced complex space with countable topology.

### 1. EXAMPLE OF A STEIN SPACE $X$ WITH $\widetilde{\mathcal{O}(X)} \neq \mathcal{O}(\tilde{X})$

Let  $(X, \mathcal{O})$  be a complex space with normalization  $\pi: \tilde{X} \rightarrow X$ . Since  $\pi$  is surjective, the map  $\pi^*: \mathcal{O}(X) \rightarrow \mathcal{O}(\tilde{X})$ ,  $f \mapsto f \circ \pi$ , is injective and the holomorphic functions  $\mathcal{O}(X)$  on  $X$  can be considered to be a subring of the holomorphic functions  $\mathcal{O}(\tilde{X})$  on the normalization  $\tilde{X}$  of  $X$ ; this will be indicated by  $\mathcal{O}(X) \subset \mathcal{O}(\tilde{X})$ . If  $X$  is irreducible and Stein, then  $\mathcal{O}(\tilde{X})$  contains the integral closure  $\widetilde{\mathcal{O}(X)}$  of  $\mathcal{O}(X)$  but does not always coincide with it, as will be shown in this section.

For an irreducible complex space  $(X, \mathcal{O})$ , the integral domain  $\mathcal{O}(X)$  is said to be *normal*, if it is integrally closed in its field of fractions  $Q(\mathcal{O}(X))$ , i.e.  $\widetilde{\mathcal{O}(X)} = \mathcal{O}(X)$ . Recall that  $Q(\mathcal{O}(X))$  is the field of meromorphic functions  $M(X)$  on  $X$  when  $X$  is irreducible and Stein due to Theorem B [10, 53.1, 52.17], and that the algebras  $M(X)$  and  $M(\tilde{X})$  are isomorphic for every complex space  $X$  [8, p. 161].

The following characterization of normal irreducible Stein spaces  $X$  by their global function algebra  $\mathcal{O}(X)$  is essentially contained in [2, § 1, p. 35].

**THEOREM 1.** *An irreducible Stein space  $X$  is normal if and only if the integral domain  $\mathcal{O}(X)$  is normal.*

An analysis of the proof shows that even when  $X$  is just irreducible and normal,  $\mathcal{O}(X)$  is also normal. Theorem 1 implies

**COROLLARY 1.** *For an irreducible Stein space  $X$  with normalization  $\tilde{X}$ , the integral closure  $\widetilde{\mathcal{O}(X)}$  of  $\mathcal{O}(X)$  is contained in  $\mathcal{O}(\tilde{X})$ .*

The following example shows that there are functions  $f \in \mathcal{O}(\tilde{X})$  which are not integral over  $\mathcal{O}(X)$ . In this example,  $X := (\mathbb{C}, \mathcal{O})$  is an irreducible

and locally irreducible Stein space given by a substructure of the canonical complex plane  $(\mathbf{C}, \mathcal{O})$ , which is then the normalization  $\tilde{X}$  of  $X$ . The substructure is defined by a "Strukturausdünnung" (see [10]) which results by replacing the stalks  $\mathcal{O}_n, n \in \mathbf{N}$ , with the stalks of generalized Neil parabolas becoming steeper as  $n$  increases. More precisely, let  $(p_n)_{n \in \mathbf{N}}$  be a strictly increasing sequence of prime numbers. For every  $n \in \mathbf{N}$ ,

$$X_n := \{(x, y) \in \mathbf{C}^2 : x^{p_n} = y^{p_n+1}\}$$

is an irreducible, locally irreducible analytic subset of  $\mathbf{C}^2$  with the origin as the only singularity and with normalization

$$\pi_n: \mathbf{C} \rightarrow X_n, t \mapsto (t^{p_n+1}, t^{p_n}).$$

Let  $f \in \mathcal{O}(\mathbf{C})$  be the identity and denote by  $\mathcal{O}_{X_n}$  the canonical complex structure on  $X_n$ . The germ  $f_0 \in \mathcal{O}_0$  of  $f$  at the origin is integral over  $\mathcal{O}_{X_n,0}$  with respect to a polynomial of degree  $p_n$ , and  $p_n$  is the minimal degree of all such polynomials.

Now define  $X := (\mathbf{C}, \mathcal{O}')$  as a substructure of the canonical plane  $(\mathbf{C}, \mathcal{O})$  with stalks

$$\mathcal{O}'_x \cong \begin{cases} \mathcal{O}_x & , x \notin \mathbf{N} \\ \mathcal{O}_{X_n,0} & , x = n \in \mathbf{N} \end{cases}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}'_n & \rightarrow & \mathcal{O}_n \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{O}_{X_n,0} & \xrightarrow{\pi_n^*} & \mathcal{O}_0, \end{array}$$

where  $\mathcal{O}'_n \rightarrow \mathcal{O}_n$  is the map induced by the identity  $(\mathbf{C}, \mathcal{O}) \rightarrow (\mathbf{C}, \mathcal{O}')$  and  $\mathcal{O}_n \cong \mathcal{O}_0$  is determined by the translation  $\mathbf{C} \rightarrow \mathbf{C}, z \mapsto z - n$ .

The identity  $f \in \mathcal{O}(\mathbf{C})$  is not integral over  $\mathcal{O}'(\mathbf{C})$ , because otherwise every polynomial of integral dependence would have degree at least  $p_n$  for all  $n \in \mathbf{N}$ .

In conclusion it should be mentioned that  $\mathcal{O}(\tilde{X})$  is almost integral over  $\mathcal{O}(X)$  [7, § 3] for every irreducible Stein space  $X$ , since  $X$  has a global universal denominator [10, E.73a].