

8. A PRIORI ESTIMATES OF ORDER FOUR

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$$\begin{aligned}
T_{3,3} + T_2 & \quad \text{when } n = 2, \\
T_{4,3} + T_{3,3,3} + T_3 & \quad \text{when } n = 3, \\
T_5 + T_{4,4} + T_4 & \quad \text{when } n = 4, \\
T_{n+1} + T_n & \quad \text{when } n \geq 5.
\end{aligned}$$

Proof. The cases $n = 2, 3, 4, 5$, must be checked bare-handed. There is no difficulty. Then, for $n \geq 5$, one can proceed by induction on n . Indeed assume,

$$\Phi_{aa'\alpha} = T_{n+1} + T_n \text{ mod. } E_{n-1}, \quad \text{for some } n = |\alpha| \geq 5.$$

Recall formula (3) and lemma 7.3; differentiating once the above equality yields

$$\Phi_{aa'\alpha b} = (T_{n+1} + T_n)_b + \Phi_{ac\alpha} \Phi_{a'c'b} = T_{n+2} + T_{n+1} \text{ mod. } E_n,$$

since $|ac\alpha| = n + 2$. The same is true with \bar{b} instead of b . Q.E.D.

Remark 7.7. The preceding lemma offers a perspective which brings some light on the type of difficulties to be expected for carrying out *a priori* estimates of each order. In particular, one may anticipate that a special step should be required for $n = 4$ (in order to kill the effect of the term $T_{4,4}$) and that the same (simpler) procedure should then apply, arguing by iteration, for any $n \geq 5$.

Notice also that the hardest case appears to be $n = 3$. Indeed, following Calabi [8] one must guess the very special *coercive* functional [1] [24]

$$S_{3,3} = \Phi_{ab'c} \Phi_{a'bc'},$$

perform a careful calculation of $\Delta'(S_{3,3})$ and use either the Maximum Principle [24] or a recurrence on $L^p(dX_{g'})$ norms of $S_{3,3}$ [1]. The *approximate* tensor calculus which we may conveniently use hereafter would not be effective for the case $n = 3$.

8. A PRIORI ESTIMATES OF ORDER FOUR

In order to prove 7.1 with $n = 4$, we consider the functional:

$$S_{4,4} = \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}} \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}} + \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}} \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}}.$$

It is enough to estimate $S_{4,4}$ since it is *coercive*. Let us compute $-\Delta'(S_{4,4})$. One readily obtains:

$$-\Delta'(S_{4,4}) = T_{6,4} + T_{5,5} \quad (\text{mod. } E_3),$$

where $T_{5,5}$ is *coercive*, while the sixth order derivatives in $T_{6,4}$ occur through $\varphi_{\bar{a}\bar{b}\alpha c c'}$ with $|\alpha| = 2$.

In view of 7.4 and 7.6, after bringing the indices cc' in first position, we get

$$(4) \quad -\Delta'(S_{4,4}) = T_{5,5} + T_{5,4} + T_{4,4,4} + T_{4,4} + T_4 \pmod{E_3}$$

where $T_{5,5}$ is the *coercive* term from above.

As expected in remark 7.7, in order to control the term $T_{4,4,4}$, we need to consider instead of $S_{4,4}$ another functional, namely:

$$\theta = S_{4,4} \exp(\varepsilon \varphi_{\bar{a}\bar{b}c} \varphi_{\bar{a}\bar{b}\bar{c}}),$$

where ε is a constant to be chosen later on. Then we compute the quantity

$$Q = -(\Delta'\theta) \exp(-\varepsilon \varphi_{\bar{a}\bar{b}c} \varphi_{\bar{a}\bar{b}\bar{c}});$$

and we easily find

$$Q = -\Delta'(S_{4,4}) + \varepsilon T_{4,4,4,4} + \varepsilon^2 T'_{4,4,4,4} + \varepsilon T_{5,4,4} \pmod{E_3},$$

where $T'_{4,4,4,4}$ is a square and where

$$T_{4,4,4,4} = S_{4,4}(\varphi_{\bar{a}\bar{b}c\bar{d}} \varphi_{\bar{a}\bar{b}\bar{c}d'} + \varphi_{\bar{a}\bar{b}c'd} \varphi_{\bar{a}\bar{b}\bar{c}d}).$$

So there exists a constant c_1 such that (see remark 5.1),

$$(S_{4,4})^2 \leq c_1 T_{4,4,4,4}.$$

Furthermore we may choose constants c_i such that,

$$\begin{aligned} |T_{5,4,4}| &\leq c_2 S_{4,4} (T_{5,5})^{\frac{1}{2}}, & |T_{5,4}| &\leq c_3 (T_{5,5} S_{4,4})^{\frac{1}{2}}, \\ |T_{4,4,4}| &\leq c_4 (S_{4,4})^{\frac{3}{2}}, & |T_{4,4}| &\leq c_5 S_{4,4}, & |T_4| &\leq c_6 (S_{4,4})^{\frac{1}{2}}. \end{aligned}$$

By splitting $T_{5,5}$ in its two halves and by putting each half together with $T_{5,4,4}$ and $T_{5,4}$ respectively, one obtains:

$$Q \geq \left(\frac{\varepsilon}{c_1} - \frac{1}{2} \varepsilon^2 c_2^2 \right) (S_{4,4})^2 - c_4 (S_{4,4})^{\frac{3}{2}} - \left(c_5 + \frac{1}{2} c_3^2 \right) S_{4,4} - c_6 (S_{4,4})^{\frac{1}{2}}.$$

Now ε must be chosen small enough in order for the coefficient of $(S_{4,4})^2$ to be *strictly* positive: $\varepsilon \in (0, (2/c_1 c_2^2))$.

To complete the proof, one argues that $Q(z_0) \leq 0$ at a point $z_0 \in X$ where θ assumes its *maximum* on X , which implies

$$S_{4,4}(z_0) \leq c_7,$$

for some controlled constant c_7 , and anywhere else on X , since $\theta \leq \theta(z_0)$ and $\|D\bar{\nabla}\bar{\nabla}\varphi\| \leq C_3$, one infers that:

$$S_{4,4} \leq c_7 \exp(2\varepsilon C_3).$$

9. A PRIORI ESTIMATES OF ORDER FIVE AND MORE

Here, in order to prove 7.1 with $n \geq 5$, we consider the functional:

$$S_{n,n} = \frac{1}{2} \sum_{|\alpha|=n-2} \varphi_{a\bar{b}\alpha} \varphi_{\bar{a}b\bar{\alpha}}$$

(the coefficient $\frac{1}{2}$ appears for both definitions of $S_{4,4}$ to agree).

Again $S_{n,n}$ is *coercive* and we compute in a similar way,

$$-\Delta'(S_{n,n}) = T_{n+2,n} + T_{n+1,n+1} \pmod{E_{n-1}},$$

where $T_{n+1,n+1}$ is *coercive*. As for $T_{n+2,n}$, proceeding as in the previous section, we find:

$$T_{n+2,n} = T_{n+1,n} + T_{n,n} + T_n \pmod{E_{n-1}}.$$

Hence,

$$-\Delta'(S_{n,n}) = T_{n+1,n+1} + T_{n+1,n} + T_{n,n} + T_n \pmod{E_{n-1}},$$

with $T_{n+1,n+1}$ *coercive*. Changing n into $(n-1)$, for $n \geq 6$, yields *still modulo* E_{n-1}

$$-\Delta'(S_{n-1,n-1}) = T'_{n,n} + T'_n \pmod{E_{n-1}}.$$

In view of formula (4) of the preceding section, this holds for $n = 5$ as well. From the *coercivity* of $T'_{n,n}$ we may choose constants $c_i > 0$, such that

$$-\Delta'(S_{n-1,n-1}) \geq c_1 S_{n,n} - c_2 (S_{n,n})^{\frac{1}{2}} - c_3.$$

Moreover we may choose constants c_i such that

$$|T_{n+1,n}| \leq 2c_4 (T_{n+1,n+1} S_{n,n})^{\frac{1}{2}}, \quad |T_{n,n}| \leq c_5 S_{n,n}, \quad |T_n| \leq c_6 (S_{n,n})^{\frac{1}{2}},$$

and $c_1 c_7 > c_4^2 + c_5$.