

# 4. Application to Systems of Algebraic Equations

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3.8. *Remark.* Let  $G$  be a Gröbner basis of an ideal  $J$ . We shall say that  $G$  is “simplified” if all  $P \in G$  fulfill the following two conditions:

$$\text{lc}(P) \text{ generates the ideal } {}_R \langle \text{lc}(Q) \mid Q \in J, \deg(Q) = \deg(P) \rangle$$

and

$$\text{in}(P) \notin \langle \text{in}(G - \{P\}) \rangle .$$

It is easy to see that the elements of a simplified Gröbner basis have pairwise different degrees.

If  $R$  is a field then  $G$  is simplified iff the elements of  $G$  have pairwise different degrees and  $\deg(G)$  is the set of minimal elements (with respect to the natural partial ordering on  $\mathbb{N}^n$ ) in  $\deg(J)$ .

If  $G$  is not simplified, then in the following way we can construct (in a finite number of steps) a simplified Gröbner basis of  $J$ :

For every  $P \in G$  choose an admissible combination  $P'$  of  $G$  such that  $\deg(P) = \deg(P')$  and  $\text{lc}(P')$  generates the ideal

$${}_R \langle \text{lc}(Q) \mid Q \in J, \deg(Q) = \deg(P) \rangle .$$

Then  $G' := \{P' \mid P \in G\}$  is a Gröbner basis of  $J$ , since  $\langle \text{in}(J) \rangle = \langle \text{in}(G) \rangle \subseteq \langle \text{in}(G') \rangle \subseteq \langle \text{in}(J) \rangle$ .

If there is a  $P' \in G'$  with  $\text{in}(P') \in \langle \text{in}(G' - \{P'\}) \rangle$ , then  $G' - \{P'\}$  is a Gröbner basis, since then  $\langle \text{in}(G' - \{P'\}) \rangle = \langle \text{in}(G') \rangle = \langle \text{in}(J) \rangle$ .

Replace  $G'$  by  $G' - \{P'\}$ . After finitely many eliminations of this kind we obtain a simplified Gröbner basis.

In example 3.7. the Gröbner basis  $F_2$  is not simplified, since  $\text{in}(P_2) = -X_2 \text{in}(P_3)$  and  $\text{in}(P_4) = 2X_2 \text{in}(P_5)$ .  $\{P_1, P_3, P_5\}$  is a simplified Gröbner basis of the ideal generated by  $F_2$ .

#### 4. APPLICATION TO SYSTEMS OF ALGEBRAIC EQUATIONS

Let  $J$  be an ideal in  $R[X]$ , generated by a subset  $F \neq \{0\}$ .

4.1. We may consider  $F$  as a system of algebraic equations in  $n$  variables. We denote by  $K$  an algebraic closure of the quotient field of  $R$ .

Let  $Z(F)$  (resp.  $Z_K(F)$ ) be the set  $\{z \in R^n$  (resp.  $K^n$ )  $\mid P(z) = 0$  for all  $P \in F\}$  of common zeros in  $R^n$  (resp.  $K^n$ ) of the elements of  $F$ . Clearly  $Z(F) = Z(J)$  and  $Z_K(F) = Z_K(J)$ .

4.2. PROPOSITION. *Let  $G$  be a Gröbner basis of  $J$ .*

1)  $Z_K(J) = \emptyset$  iff  $G \cap R \neq \emptyset$ .

2) *The set  $Z_K(J)$  is finite iff  $\mathbb{N}^n - \mathcal{D}(G)$  is finite. In this case the cardinality of  $Z_K(J)$  is smaller than or equal to the cardinality of  $\mathbb{N}^n - \mathcal{D}(G)$ .*

*Proof.*

1) By Hilbert's Nullstellensatz we know:

$Z_K(J) = \emptyset$  iff  $J \cap R \neq \emptyset$ . Therefore  $Z_K(J) = \emptyset$  implies  $0 \in \text{deg}(J)$ , hence  $G \cap R \neq \emptyset$ .

2) Let  $I$  be the ideal generated by  $J$  in  $K[X]$ . Then  $F$  is a Gröbner basis of  $I$ , too. Again by Hilbert's Nullstellensatz the dimension (as  $K$ -vector space) of  $K[X]/I$  is an upper bound for the cardinality of  $Z_K(J) = Z_K(I)$ , and this dimension is finite iff  $Z_K(J)$  is so. Since  $G$  is a Gröbner basis of  $I$ , one easily verifies that the residue classes  $X^\alpha + I$ ,  $\alpha \in \mathbb{N}^n - \mathcal{D}(G)$ , form a  $K$ -basis of  $K[X]/I$ . This proves the proposition.

4.3. PROPOSITION. *Let  $G$  be a Gröbner basis of  $J$  with respect to the lexicographic ordering (see 1.2.).*

*If  $J \cap R[X_k, \dots, X_n] \neq \{0\}$ , then*

$$G_k := G \cap R[X_k, \dots, X_n]$$

*is a Gröbner basis of*

$$J_k := J \cap R[X_k, \dots, X_n];$$

*in particular,  $G_k$  generates the ideal  $J_k \leq R[X_k, \dots, X_n]$  ( $1 \leq k \leq n$ ).*

*Proof.* Let  $Q \in J_k$ . For any  $P \in R[X]$  with  $\text{deg}(P) \leq \text{deg}(Q)$  we have  $P \in R[X_k, \dots, X_n]$ , since  $<$  is the lexicographic ordering. By 2.2. and 2.5. there are  $c(\alpha, P) \in R$  such that  $Q = \sum_{P \in G, \alpha \in \mathbb{N}^n} c(\alpha, P) X^\alpha P$  and  $c(\alpha, P) \neq 0$  implies  $\text{deg}(X^\alpha P) \leq \text{deg}(Q)$ .

Hence we have  $X^\alpha P \in R[X_k, \dots, X_n]$  for  $c(\alpha, P) \neq 0$ , and, by 2.5. again,  $G_k$  is a Gröbner basis of  $J_k$ .

4.4. Now we can apply the theory of Gröbner bases to find the solutions to the system  $F$  of algebraic equations. Consider the following algorithm:

First we construct a Gröbner basis  $G$  of  $J$  with respect to the lexicographic ordering (see 3.6.). As in 4.3. we write  $G_k$  for  $G \cap R[X_k, \dots, X_n]$ ,  $1 \leq k \leq n$ .

Compute the greatest common divisor  $P_n$  of the (univariate) polynomials in  $G_n$ . Find a zero  $a_n \in R$  of  $P_n$ . If  $P_n$  has no zero in  $R$ , then  $Z(J) = \emptyset$ .

Let  $k \in \{1, \dots, n-1\}$ . Suppose that  $a_{k+1}, \dots, a_n \in R$  have already been found. Let  $G_k(a_{k+1}, \dots, a_n) \subseteq R[X_k]$  be the set of polynomials in one variable  $X_k$  obtained from  $G_k$  by substituting everywhere  $a_j$  for  $X_j$ ,  $k+1 \leq j \leq n$ .

Compute the greatest common divisor  $P_k$  of the polynomials in  $G_k(a_{k+1}, \dots, a_n)$ . Find a zero  $a_k \in R$  of  $P_k$ . If  $P_k$  has no zero in  $R$ , we have to go back to  $G_n$  and to find another sequence  $a'_n, \dots, a'_{k+1}$ .

If we obtain  $(a_1, \dots, a_n)$  by this algorithm, it is an element of  $Z(J)$ . By 4.3. all elements of  $Z(J)$  can be computed in this way.

Suppose that  $Z_K(J)$  is finite (i.e.  $\mathbf{N}^n - \mathcal{D}(G)$  is finite) and that we are able to solve univariate polynomial equations in  $R$  (which is the case for  $R = \mathbf{Z}$ ). Then the algorithm above yields  $Z(J)$  in a finite number of steps.

4.5. *Example.* Let  $F$  be the subset

$$\begin{aligned} &\{2X_1^4 + 3X_1^3X_2X_3 - X_1X_2^2 + 5X_1 - 3X_2^2 - 5X_2X_3 - 2X_3 + 41, \\ &4X_1^4 + 6X_1^3X_2X_3 - 2X_1X_2^2 + 10X_1 + 3X_2^2 + 5X_2X_3 + 2X_3^3 - 11X_3^2 + 19X_3 + 25, \\ &6X_2^2 + 10X_2X_3 + 2X_3^3 - 11X_3^2 + 21X_3 - 40\} \quad \text{of} \quad \mathbf{Z}[X_1, X_2, X_3]. \end{aligned}$$

By the algorithm 3.6. we get a Gröbner basis  $G$  of the ideal generated by  $F$ :

$$\begin{aligned} G = &\{2X_3^3 - 11X_3^2 + 17X_3 - 6, \\ &3X_2^2 + 5X_2X_3 + 2X_3 - 17, \\ &2X_1^4 + 3X_1^3X_2X_3 - X_1X_2^2 + 5X_1 + 24\}. \end{aligned}$$

Now  $Z(G_3) = \{2, 3\}$ ,  $Z(G_2(2)) = \{1\}$ ,  $Z(G_2(3)) = \emptyset$  and  $Z(G_1(1, 2)) = \{-2\}$ . So  $Z(F) = \{(-2, 1, 2)\}$ .

## 5. APPLICATION TO A GEOMETRIC PROBLEM

5.1. For  $P \in R[X]$  let  $\tilde{P}$  be the homogeneization of  $P$  by a further variable  $X_{n+1}$ . For an ideal  $J \subseteq R[X]$  we write  $\tilde{J}$  for the ideal generated by  $\{\tilde{P} \mid P \in J\}$  in  $R[X_1, \dots, X_{n+1}]$ .

PROPOSITION. Let  $G$  be a Gröbner basis of  $J$  with respect to the graded inverse lexicographic ordering (see 2.1.). Then  $\tilde{G} := \{\tilde{P} \mid P \in G\}$  is a Gröbner basis of  $\tilde{J}$ .