

4. Properness

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LEMMA 2.1. Let A, B be metric spaces, with $A \neq \emptyset$ and B connected. Let $P: A \rightarrow B$ be a continuous map. Assume:

- (i) P is open,
- (ii) P is proper, that is, for any compact subset K in B , $P^{-1}(K)$ is compact. Then P is surjective.

Proof. We only need to prove that $P(A)$ is closed. Let b_0 be a point in $\overline{P(A)}$. Since B is a metric space, there exists a sequence $(b_i)_{i>0}$ in $P(A)$ converging to b_0 . The subset $K = \{b_0, b_1, b_2, \dots\}$ is compact, hence so is $PP^{-1}(K)$. The latter contains $b_1, \dots, b_i, \dots$, hence b_0 , and it is obviously contained in $P(A)$. Q.E.D.

In order to make use of this lemma, we shall need some inverse function theorem for (i), and some *a priori* estimates for (ii).

3. LOCAL INVERSION

THEOREM 3.1. Let X be a smooth compact manifold, V and W smooth vector bundles on X , U an open set in $C^\infty(X, V)$, and $P: U \rightarrow C^\infty(X, W)$, a smooth nonlinear elliptic partial differential operator. Let A and B be LCFC submanifolds of U and of $C^\infty(X, W)$ respectively, such that the restriction P_A of P to A , sends A into B . Then the Jacobian criterion holds for P_A , namely, if the derivative of $P_A: A \rightarrow B$ is invertible at $\varphi_0 \in A$, then P_A is a local diffeomorphism near φ_0 .

This is a convenient variant of the Nash-Moser theorem (e.g. [14]) regarding suitable restrictions of elliptic operators. It is established in a separate paper [11] (see also [22]). It relies only on the classical (Banach) inverse function theorem combined with *elliptic regularity*.

Remark 3.2. The Nash-Moser theorem has been studied by many authors, see the bibliography below and further references in [14] [15] [25].

4. PROPERNESS

In view of (2), theorem 3.1 implies that P_λ is open. We want to apply lemma 2.1 in order to prove that P_λ is surjective from A_λ to B_λ . Since $P_\lambda(A_\lambda) \neq \emptyset$ (it contains 0), and since B_λ is connected, this amounts to proving that P_λ is *proper*. Let us explain why *a priori* estimates imply properness.

Concerning subsets in A_λ we have

PROPOSITION 4.1. *A subset S in A_λ is relatively compact in A_λ iff its closure $\bar{S}$ in $C^\infty(X)$ lies inside A_λ and S is bounded in $C^\infty(X)$.*

This readily follows from Ascoli theorem which implies the well-known fact [12] (p. 231) that in $C^\infty(X)$ (and in any closed LCFC submanifold of $C^\infty(X)$, such as B_λ , as well) bounded subsets are relatively compact and vice-versa; hence, compact subset of A_λ are nothing but bounded closed strictly interior subsets of A_λ . Explicitely, let us state the

COROLLARY 4.2. *A closed subset S in A_λ is compact if and only if there exists a sequence $(C_i), i \in \mathbb{N}$, of positive numbers, such that for any φ in S the following estimates hold:*

$$\begin{aligned} \|(g')^{-1}\| &=: \sup_X |(g')^{-1}| \leq C_0, \\ \forall i \in \mathbb{N}, \quad \|D^i \varphi\| &=: \sup_X |D^i \varphi| \leq C_i, \end{aligned}$$

where $|\cdot|$ denotes some natural norms of tensors in the original metric g , and $D =: (\nabla, \bar{\nabla})$ is the total covariant differentiation with respect to the metric g .

Proof. Indeed S is closed and bounded. Moreover, since for $\varphi \in S$,

$$\|(g')^{-1}\| \leq C_0$$

all the eigenvalues of $(g')^{-1}$ (which are positive) are uniformly bounded from above, hence those of g' are uniformly bounded from below, in other words:

$$\exists \varepsilon > 0, \quad \forall \varphi \in S, \quad g' \geq \varepsilon g,$$

or equivalently $\bar{S}$ lies strictly inside A_λ . Q.E.D.

In the next sections we will show that if f belongs to some compact (i.e. bounded and closed) subset K of B_λ , defined by a sequence $(K_i), i \in \mathbb{N}$, such that $\|D^i f\| \leq K_i$, then for $\varphi \in A_\lambda$ satisfying $P_\lambda(\varphi) = f$, the following *a priori* estimates hold:

$$\|\varphi\| \leq C_0, \quad \forall i \in \mathbb{N}, \quad \|D^i \nabla \bar{\nabla} \varphi\| \leq C_{i+2}.$$

These estimates imply that P_λ is proper, i.e. that $S = P_\lambda^{-1}(K)$ is compact, according to the following

PROPOSITION 4.3. *Let S be a closed subset in A_λ . Suppose that there exists a sequence $(C_i), i \in \mathbb{N}$, such that for any φ in S , the following estimates hold:*

$$\| \varphi \| \leq C_0, \quad \| P_\lambda(\varphi) \| \leq C_0, \quad \forall i \in \mathbf{N}, \quad \| D^i \nabla \bar{\nabla} \varphi \| \leq C_{i+2}.$$

Then S is compact.

Proof. The first two estimates imply a uniform estimate

$$| \operatorname{Log} \det (g' g^{-1}) | \leq E.$$

The estimate on $\| \nabla \bar{\nabla} \varphi \|$ yields another one:

$$\| g' \| \leq F.$$

These two estimates yield

$$\| (g')^{-1} \| \leq G.$$

Now from $\| D^i \nabla \bar{\nabla} \varphi \| \leq C_{i+2}$ we infer

$$\| D^i \Delta \varphi \| \leq \tilde{C}_{i+2}$$

since D and g^{-1} commute (Δ denotes the Laplacian in the metric g). As Δ performs a continuous linear automorphism of the Fréchet space of smooth functions *with zero average* (by Fredholm theory), the Closed Graph Theorem implies the missing estimates. Q.E.D.

Remark 4.4. Actually we have been considering two *gradings* of $C^\infty(X)$ [14]. The usual one, namely the one defined, $\forall u \in C^\infty(X)$, by

$$\| u \|_0 = \sup_X | u |,$$

$$\| u \|_i = \| u_i \|_{i-1} + \| D^i u \|, \quad i \geq 1,$$

and another one, well-adapted here since the true unknown is a Kähler metric, defined by

$$\| u \|_0^* = \| u \|_0, \quad \| u \|_1^* = \| u \|_1,$$

$$\| u \|_i^* = \| u \|_{i-1}^* + \| D^{i-2}(\nabla \bar{\nabla} u) \|, \quad i \geq 2.$$

Although it is unnecessary for the purpose of proposition 4.3, it can be shown globally (without Schauder theory) that these two gradings are *tamely* equivalent [14] of degree 2 and base 0 [10] (section 5). Hence, they define the same topology.