

2. SOME FURTHER RESULTS

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Therefore, $f(\Psi_n)$ is the set Φ_n of mutually isoclinic n -planes in our Theorem 1.6.

2. SOME FURTHER RESULTS

From now on we shall confine our attention to n -dimensional maximal sets of mutually isoclinic n -planes in R^{2n} , and therefore, n has always the values 2, 4, or 8 unless stated otherwise.

In this section, we prove a few more theorems for use in § 3. In these theorems, the indices a, b have the range of values $(0, 1, \dots, n-1)$; $B_0 = I$ is the identity matrix of order n ; B_1, \dots, B_{n-1} are the $n \times n$ matrices listed in Theorems 1.5 and 1.6; $\lambda = (\lambda_a)$ is an ordered set of n real parameters; and

$$B(\lambda) \equiv \sum_a \lambda_a B_a, \quad N(\lambda) \equiv \sum_a \lambda_a^2.$$

Moreover, for any matrix M , we denote its transpose by M^T .

THEOREM 2.1.

- (i) $B(\lambda)B(\lambda)^T = N(\lambda)I$.
- (ii) If $\lambda \neq 0$, then

$$B(\lambda)^{-1} = B(\lambda)^T/N(\lambda) = \sum_a \lambda_a B_a^T/N(\lambda),$$

so that if $\lambda \neq 0$, the equation $y = xB(\lambda)$ is equivalent to the equation $x = yB(\mu)^T$, where $\mu = \lambda/N(\lambda) \neq 0$.

$$(iii) \quad \det B(\lambda) = + (N(\lambda))^{n/2}.$$

(iv) If $N(\lambda) = 1$, then $B(\lambda) \in SO(n)$, where $SO(n)$ is the set of all orthogonal matrices of order n and determinant $+1$.

$$\begin{aligned} \text{Proof.} \quad B(\lambda)B(\lambda)^T &= (\sum_a \lambda_a B_a) (\sum_b \lambda_b B_b^T) = \sum_{a,b} \lambda_a \lambda_b B_a B_b^T \\ &= \sum_a \lambda_a^2 B_a B_a^T + \sum_{a < b} \lambda_a \lambda_b (B_a B_b^T + B_b B_a^T), \end{aligned}$$

which, on account of the Hurwitz matrix equations (1.2), is equal to $(\sum_a \lambda_a^2)I = N(\lambda)I$. This proves (i), and also (ii). To prove (iii), we first note that since $B(\lambda)$ is a square matrix of order n , $\det B(\lambda)$ is a homogeneous polynomial of degree n in the λ_a 's, and it follows from (i) that

$$(\det B(\lambda))^2 = \det (B(\lambda)B(\lambda)^T) = (N(\lambda))^n.$$

Therefore,

$$(2.1) \quad \det B(\lambda) = \pm (N(\lambda))^{n/2} = \pm (\lambda_0^2 + \lambda_1^2 + \dots + \lambda_{n-1}^2)^{n/2} \\ = \pm (\lambda_0^n + \text{other product terms in } \lambda_a).$$

On the other hand, since $B_0 = I$, and B_1, \dots, B_{n-1} are all skew-symmetric matrices, the diagonal elements of $B(\lambda)$ are all equal to λ_0 , and none of the other elements of $B(\lambda)$ is equal to λ_0 . Therefore,

$$\det B(\lambda) = \lambda_0^n + \text{other product terms in } \lambda_a.$$

Comparison of this with (2.1) gives (iii). Finally, (iv) follows immediately from (i) and (iii).

Returning to Theorems 1.2 and 1.6, we now prove

THEOREM 2.2. *Let Φ_n be the maximal set of mutually isoclinic n -planes in R^{2n} described in Theorem 1.6, and let (u, v) be any vector in R^{2n} . If $u \neq 0$, then the unique n -plane in Φ_n containing (u, v) is*

$$(2.2) \quad y = x[vu^T - (vB_1u^T)B_1 - \dots - (vB_{n-1}u^T)B_{n-1}]/(uu)^T.$$

If $v \neq 0$, then the unique n -plane in Φ_n containing (u, v) is

$$(2.3) \quad x = y[uv^T - (uB_1^T v^T)B_1^T - \dots - (uB_{n-1}^T v^T)B_{n-1}^T]/(vv)^T.$$

Here, B_1, \dots, B_{n-1} are the matrices in (1.3), (1.4), or (1.5) according as $n = 2, 4$, or 8 .

Proof. We shall prove only (2.2) for the case $u \neq 0$, as (2.3) for the case $v \neq 0$ can be proved similarly. Suppose that $u \neq 0$ and

$$(2.4) \quad y = x(\lambda_0 + \lambda_1 B_1 + \dots + \lambda_{n-1} B_{n-1})$$

is an n -plane in Φ_n containing (u, v) . Then we have

$$v = u(\lambda_0 + \lambda_1 B_1 + \dots + \lambda_{n-1} B_{n-1}),$$

which can be written as

$$v = [\lambda_0 \lambda_1 \dots \lambda_{n-1}] \begin{bmatrix} u \\ uB_1 \\ \vdots \\ uB_{n-1} \end{bmatrix}.$$

Multiplying the two sides of this equation on the right by

$$[u^T, -B_1 u^T, \dots, -B_{n-1} u^T]$$

and making use of the Hurwitz matrix equations (1.2), we get

$$v[u^T, -B_1 u^T, \dots, -B_{n-1} u^T] = [\lambda_0 \lambda_1 \dots \lambda_{n-1}] (uu^T)I.$$

Since $uu^T \neq 0$, the above equation determines the λ_a 's uniquely in terms of u, v . Now with these values of λ_a 's, equation (2.4) becomes equation (2.2), as we wanted to prove. Incidentally, the above proof also confirms that there is exactly one n -plane in Φ_n containing the vector (u, v) (cf. Theorem 1.2).

Next, we give a direct proof of Theorem 1.3 for the special cases $n = 2, 4, 8$, and state the result as

THEOREM 2.3. *The maximal set $\Phi_n = \{x = 0, y = xB(\lambda)\}$ of mutually isoclinic n -planes in R^{2n} , $n = 2, 4$, or 8 , can be given a differentiable structure so that it is diffeomorphic with the n -sphere S^n .*

Proof. Let us regard Φ_n as a point set whose elements are the n -planes in Φ_n . Then, the subset $\Phi_n \setminus \mathbf{O}^\perp = \{y = xB(\lambda)\}$ of Φ_n is an open subset in which we can define a coordinate system by assigning to the element $y = xB(\lambda)$ the coordinate $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{n-1})$. The subset $\Phi_n \setminus \mathbf{O} = \{x = 0 \text{ and } y = xB(\lambda), \text{ where } \lambda \neq 0\}$ of Φ_n is also an open subset. By Theorem 2.1 (ii), this subset is the same as the subset $\{x = yB(\mu)^T\}$, and so, we can define in it a coordinate system by assigning to the element $x = yB(\mu)^T$ the coordinate $\mu = (\mu_0, \mu_1, \dots, \mu_{n-1})$. Thus Φ_n is covered by the two coordinate neighborhoods

$$(2.5) \quad (\Phi_n \setminus \mathbf{O}^\perp, \lambda), \quad (\Phi_n \setminus \mathbf{O}, \mu).$$

Moreover, we can see from Theorem 2.1 (ii) that for any element in $(\Phi_n \setminus \mathbf{O}^\perp) \cap (\Phi_n \setminus \mathbf{O}) = \Phi_n \setminus \{\mathbf{O}^\perp, \mathbf{O}\}$, its two coordinates λ, μ , both nonzero, are related by

$$(2.6) \quad \mu = \lambda/N(\lambda), \text{ or equivalently, } \lambda = \mu/N(\mu).$$

Hence, Φ_n is an n -dimensional manifold.

To show that Φ_n is diffeomorphic with the n -sphere S^n , we view S^n as the unit sphere $x_1^2 + \dots + x_{n+1}^2 = 1$ in R^{n+1} , and use stereographic projections. Let $q_1(0, \dots, 0, 1)$ and $q_2(0, \dots, 0, -1)$ be respectively the north and south poles of S^n . Then S^n is the union of the two open subsets

$S^n \setminus q_1$ and $S^n \setminus q_2$. For an arbitrary point q in $S^n \setminus q_1$, let the line $q_1 q$ meet the equator n -plane $x_{n+1} = 0$ at the point $(\lambda, 0)$; and for an arbitrary point q in $S^n \setminus q_2$, let the line $q_2 q$ meet the equator n -plane $x_{n+1} = 0$ at the point $(\mu, 0)$. Then S^n is covered by the two coordinate neighborhoods

$$(2.7) \quad (S^n \setminus q_1, \lambda), \quad (S^n \setminus q_2, \mu).$$

Moreover, it is easy to verify that for a point in $S^n \setminus \{q_1, q_2\}$, its two coordinates λ and μ are also both nonzero and related by (2.6).

It now follows from (2.5), (2.6) and (2.7) that if f_1 is the map from $\Phi_n \setminus \mathbf{O}^\perp$ to $S^n \setminus q_1$ sending an n -plane in $\Phi_n \setminus \mathbf{O}^\perp$ with coordinate λ to the point in $S^n \setminus q_1$ with the same coordinate λ , and f_2 is the map from $\Phi_n \setminus \mathbf{O}$ to $S^n \setminus q_2$ sending an n -plane in $\Phi_n \setminus \mathbf{O}$ with coordinate μ to the point in $S^n \setminus q_2$ with the same coordinate μ , then f_1, f_2 combined will give a diffeomorphism from Φ_n to S^n .

In the remainder of this section, we are concerned exclusively with the matrices $B(\lambda)$ with $N(\lambda) = 1$. For convenience, we shall denote such matrices by $B(\lambda')$, with the understanding that λ' always satisfies the condition $N(\lambda') = 1$.

We know from Theorem 2.1 (iv) that every $B(\lambda')$ belongs to $SO(n)$. Let us now regard $SO(n)$ as the special orthogonal group. Then the set of elements $B(\lambda')$ of $SO(n)$ will generate a subgroup of $SO(n)$. We wish to know what this subgroup of $SO(n)$ is, and the next three theorems will give us the answer.

THEOREM 2.4. *For $n = 2$, the set of elements $B(\lambda')$ forms the group $SO(2)$ which is isomorphic with S^1 .*

Proof. Since

$$B(\lambda') = \begin{bmatrix} \lambda'_0 & \lambda'_1 \\ -\lambda'_1 & \lambda'_0 \end{bmatrix} \quad \text{and} \quad \det B(\lambda') = (\lambda'_0)^2 + (\lambda'_1)^2 = 1,$$

the elements of $SO(2)$ are the elements $B(\lambda')$ themselves.

THEOREM 2.5. *For $n = 4$, the set of elements $B(\lambda')$ forms a 3-parameter subgroup of $SO(4)$, isomorphic with S^3 .*

Proof. First, since $N(\lambda') = (\lambda'_0)^2 + \dots + (\lambda'_3)^2 = 1$, the set $B(\lambda')$, with a natural topology, is homeomorphic with the unit 3-sphere S^3 in R^4 . Next, using (1.4), we can easily verify that

$$B_2 B_3 = -B_1, \quad B_3 B_1 = -B_2, \quad B_1 B_2 = -B_3.$$

With this and Theorem 2.1 (ii), straight forward computation will show that for any two elements $B(\lambda')$ and $B(\mu')$ of $SO(4)$, the product $B(\lambda')B(\mu')^{-1}$ is an element of $SO(4)$ of the form $B(v')$, where the components of v' are analytic functions of the components of λ' and μ' . This proves our theorem.

For the case $n = 8$, we first observe that the elements $B(\lambda')$ of $SO(8)$ do not, by themselves, form a subgroup of $SO(8)$. For example, although B_1, B_2 are both of the form $B(\lambda')$, their product $B_1 B_2$ is not. In fact, we have

THEOREM 2.6. *For $n = 8$, the set of elements $B(\lambda')$ of $SO(8)$ generates the group $SO(8)$ itself.*

Proof. Our proof consists of two steps (i) and (ii). In (i), we prove that the 28 skew-symmetric 8×8 matrices $B_i, B_i B_j (i, j = 1, \dots, 7, \text{ and } i < j)$ are linearly independent. In (ii), we prove that the Lie algebra of the subgroup of $SO(8)$ generated by the elements $B(\lambda')$ coincides with the Lie algebra $\mathfrak{o}(8)$ of $SO(8)$. The assertion in our theorem then follows from the well-known fact in Lie groups that there is a one-one correspondence between the connected Lie subgroups of a Lie group G and the Lie subalgebras of the Lie algebra of G .

(i) From (1.5), we see that the 8×8 matrices $B_i (i = 1, \dots, 7)$ can be partitioned as

$$B_1 = \begin{bmatrix} J & & & \\ & J & & \\ & & J & \\ & & & -J \end{bmatrix}, \quad B_2 = \begin{bmatrix} & K & & \\ -K & & & \\ & & I & \\ & & & -I \end{bmatrix}, \quad B_3 = \begin{bmatrix} & & L & \\ & & & \\ -L & & & \\ & & & J \end{bmatrix}$$

$$B_4 = \begin{bmatrix} & K & & \\ & & -I & \\ -K & & & \\ & I & & \end{bmatrix}, \quad B_5 = \begin{bmatrix} & & L & \\ & & & -J \\ -L & & & \\ & -J & & \end{bmatrix},$$

$$B_6 = \begin{bmatrix} & & I \\ & K & \\ -I & -K & \end{bmatrix}, \quad B_7 = \begin{bmatrix} & & J \\ & L & \\ J & -L & \end{bmatrix},$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

are 2×2 submatrices and each empty space represents a 2×2 zero-matrix 0.

Since the matrices I, J, K, L have the properties

$$I^2 = I, \quad J^2 = -I, \quad K^2 = I, \quad L^2 = I, \\ JK = -KJ = -L, \quad KL = -LK = J, \quad LJ = -JL = -K,$$

we can easily verify that the products $B_i B_j (i, j = 1, \dots, 7, \text{ and } i < j)$ are matrices of the same form as B_i , having some of $O, \pm I, \pm J, \pm K, \pm L$ as 2×2 submatrices.

To prove that the 28 matrices $B_i, B_i B_j$ are linearly independent, we construct the 8×8 matrix

$$M \equiv \sum_i \alpha_i B_i + \sum_{i < j} \alpha_{ij} (B_i B_j),$$

where the α 's are some real numbers, and show that if $M = 0$, then all the α 's are zero. Let $M = [M_{hk}]$, where $M_{hk} (h, k = 1, 2, 3, 4)$ are the 2×2 submatrices of M . Then by using the explicit forms of B_i and $B_i B_j$, we can write M as the sum of the following four matrices:

$$\begin{bmatrix} M_{11} & & & \\ & M_{22} & & \\ & & M_{33} & \\ & & & M_{44} \end{bmatrix} = \alpha_1 \begin{bmatrix} J & & & \\ & J & & \\ & & J & \\ & & & -J \end{bmatrix} + \alpha_{23} \begin{bmatrix} -J & & & \\ & -J & & \\ & & J & \\ & & & -J \end{bmatrix} \\ + \alpha_{45} \begin{bmatrix} -J & & & \\ & J & & \\ & & -J & \\ & & & -J \end{bmatrix} + \alpha_{67} \begin{bmatrix} J & & & \\ & -J & & \\ & & -J & \\ & & & -J \end{bmatrix},$$

$$\begin{bmatrix} & M_{12} \\ M_{21} & \\ & M_{34} \\ & M_{43} \end{bmatrix} = \alpha_2 \begin{bmatrix} & K \\ -K & \\ & I \\ & -I \end{bmatrix} + \alpha_{13} \begin{bmatrix} & K \\ -K & \\ & \\ & -I \\ & I \end{bmatrix} \\
 + \alpha_3 \begin{bmatrix} & L \\ -L & \\ & J \\ & J \end{bmatrix} + \alpha_{12} \begin{bmatrix} & -L \\ L & \\ & \\ & J \\ & J \end{bmatrix} \\
 + \alpha_{46} \begin{bmatrix} & -I \\ I & \\ & -K \\ & K \end{bmatrix} + \alpha_{57} \begin{bmatrix} & -I \\ I & \\ & \\ & K \\ & -K \end{bmatrix} \\
 + \alpha_{47} \begin{bmatrix} & -J \\ -J & \\ & -L \\ & L \end{bmatrix} + \alpha_{56} \begin{bmatrix} & J \\ J & \\ & \\ & -L \\ & L \end{bmatrix},$$

$$\begin{bmatrix} & M_{13} \\ M_{31} & \\ & M_{24} \\ & M_{42} \end{bmatrix} = \alpha_4 \begin{bmatrix} & K \\ -K & \\ & -I \\ & I \end{bmatrix} + \alpha_{15} \begin{bmatrix} & K \\ -K & \\ & \\ & I \\ & -I \end{bmatrix} \\
 + \alpha_5 \begin{bmatrix} & L \\ -L & \\ & -J \\ & -J \end{bmatrix} + \alpha_{14} \begin{bmatrix} & -L \\ L & \\ & \\ & -J \\ & -J \end{bmatrix} \\
 + \alpha_{26} \begin{bmatrix} & I \\ -I & \\ & -K \\ & K \end{bmatrix} + \alpha_{37} \begin{bmatrix} & I \\ -I & \\ & \\ & K \\ & -K \end{bmatrix} \\
 + \alpha_{27} \begin{bmatrix} & J \\ J & \\ & -L \\ & L \end{bmatrix} + \alpha_{36} \begin{bmatrix} & -J \\ -J & \\ & \\ & -L \\ & L \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} & & & M_{14} \\ & & M_{23} & \\ & M_{32} & & \\ M_{41} & & & \end{bmatrix} &= \alpha_6 \begin{bmatrix} & & I \\ & K & \\ -I & -K & \end{bmatrix} + \alpha_{17} \begin{bmatrix} & & -I \\ & K & \\ I & -K & \end{bmatrix} \\
&+ \alpha_7 \begin{bmatrix} & J \\ & L \\ J & -L \end{bmatrix} + \alpha_{16} \begin{bmatrix} & J \\ & -L \\ J & L \end{bmatrix} \\
&+ \alpha_{24} \begin{bmatrix} & -K \\ & -I \\ K & I \end{bmatrix} + \alpha_{35} \begin{bmatrix} & K \\ & -I \\ -K & I \end{bmatrix} \\
&+ \alpha_{25} \begin{bmatrix} & -L \\ & -J \\ L & -J \end{bmatrix} + \alpha_{34} \begin{bmatrix} & -L \\ & J \\ L & J \end{bmatrix}.
\end{aligned}$$

Now, $M = 0$ means that all its submatrices M_{hk} are zero. Since I, J, K, L are linearly independent, the equations $M_{hk} = 0$ are equivalent to a number of linear equations in the α 's, and from these linear equations we can easily see that the α 's must all be zero. For example, it is obvious from the equations

$$M_{12} = (\alpha_2 + \alpha_{13})K + (\alpha_3 - \alpha_{12})L - (\alpha_{46} + \alpha_{57})I - (\alpha_{47} - \alpha_{56})J = 0,$$

$$M_{34} = (\alpha_2 - \alpha_{13})I + (\alpha_3 + \alpha_{12})J + (-\alpha_{46} + \alpha_{57})K - (\alpha_{47} + \alpha_{56})L = 0$$

that

$$\alpha_2, \quad \alpha_{13}, \quad \alpha_3, \quad \alpha_{12}, \quad \alpha_{46}, \quad \alpha_{57}, \quad \alpha_{47}, \quad \alpha_{56}$$

must all be zero. Thus we have proved that the 28 matrices $B_i, B_i B_j$ are linearly independent.

(ii) Let G be the Lie subgroup of $SO(8)$ generated by the elements $B(\lambda')$, and g its Lie algebra. Then g is a Lie subalgebra of the Lie algebra $\mathfrak{o}(8)$ of $SO(8)$. We now prove that in fact $g = \mathfrak{o}(8)$.

From the theory of Lie groups we know that if $t \rightarrow f(t)$, where $t \in \mathbb{R}$ and $f(t) \in G$, is any curve in G passing through the identity element

$I = f(0)$ of G , then the velocity vector $f'(0)$ of this curve at I is an element of g . Now

$$t \rightarrow f_i(t) \equiv (\cos t)I + (\sin t)B_i \quad (i=1, \dots, 7)$$

are obviously curves in G such that $f_i(0) = I$ and $f'_i(0) = B_i$. Therefore, B_i are all elements of g .

Since g is a Lie subalgebra of $o(8)$ and $B_i \in g$, the Lie products $[B_i, B_j] = B_i B_j - B_j B_i = 2B_i B_j$, where $i, j = 1, \dots, 7$, and $i < j$, are all in g .

We have thus proved that the 28 linearly independent skew-symmetric matrices, $B_i, B_i B_j$ all belong to $g \subset o(8)$. Since $o(8)$ is the Lie algebra of all skew-symmetric matrices of order 8 and is therefore of dimension 28, g coincides with $o(8)$. This completes the proof of Theorem 2.6.

3. THE SPHERE BUNDLES $S^{2n-1} \rightarrow \Phi_n$, $n = 2, 4$, OR 8 , WITH FIBERS ON MUTUALLY ISOCLINIC n -PLANES IN R^{2n}

In R^{2n} , $n = 2, 4$, or 8 , provided with rectangular coordinate system (x, y) , let S^{2n-1} be the unit sphere and Φ_n the maximal set of mutually isoclinic n -planes $\{x = 0, y = xB(\lambda)\}$ defined in Theorem 1.6. Then with the preparations we have made in § 2, we can now prove

THEOREM 3.1. *In R^{2n} , $n = 2, 4$, or 8 , the n -planes in the maximal set Φ_n of mutually isoclinic n -planes slice the unit sphere S^{2n-1} into a fiber bundle*

$$\mathcal{J}_n = (S^{2n-1}, \Phi_n, \pi, S^{n-1}, G_n),$$

with base space Φ_n , projection π , fiber S^{n-1} and group G_n , where $G_2 = S^1$, $G_4 = S^3$, and $G_8 = SO(8)$.

Proof. We prove by exhibiting all the ingredients of a representative coordinate bundle.

(1) The bundle space S^{2n-1} has the equation $xx^T + yy^T = 1$ in R^{2n} .

(2) The base space Φ_n is covered by the two coordinate systems

$$(2.5) \quad (\Phi_n \setminus \mathbf{O}^\perp, \lambda), \quad (\Phi_n \setminus \mathbf{O}, \mu)$$

as in the proof of Theorem 2.3, where \mathbf{O}^\perp is the n -plane $x = 0$, \mathbf{O} is the n -plane $y = 0$, λ is the parameter in the equation $y = xB(\lambda)$ of an n -plane in $\Phi_n \setminus \mathbf{O}^\perp$, and μ is the parameter in the equation $x = yB(\mu)^T$ of