5. COMPARISON OF OUR BUNDLES WITH THE HOPF-STEENROD BUNDLES

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **28.03.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

For $n = 8$, the X, Y, C and D are Cayley numbers. On putting $X = c$, $Y = d$, $C = x$ and $D = y$, we can see immediately that the projection (4.9′) and the coordinate functions (4.10′) and (4.11′) are exactly those of the linear bundle $\mathscr{B}$ constructed by N. Steenrod in [5, pp. 109-110]. Therefore, this linear bundle $\mathscr{B}$ of Steenrod and the linear bundle $\mathscr{SL}_8$ in Theorem 4.2 are two slightly different representations of the same bundle.

5. Comparison of our bundles with the Hopf-Steenrod bundles

In § 3, we constructed the sphere bundles

$$\mathscr{I}_n = (S^{2n-1}, \Phi_n, \pi, S^{n-1}, G_n), \quad n = 2, 4, 8,$$

with fibers lying on mutually isoclinic n-planes in R^{2n}. In § 4, we gave a unified treatment of the classical Hopf-Steenrod sphere bundles

$$\mathscr{HS}_n = (S^{2n-1}, S^n, p, S^{n-1}, O(n)), \quad n = 2, 4, 8,$$

using, as N. Steenrod did, the Hopf map and the hypercomplex systems of complex numbers, quaternions and Cayley numbers. In this section we shall prove that (i) the Hopf fibering $S^{2n-1} \to S^n$ and our maximal set of mutually isoclinic n-planes in R^{2n} are equivalent concepts (Theorems 5.1 and 5.2), and (ii) the representative coordinate bundles constructed in § 3 and § 4 for the bundles $\mathscr{I}_n$ and $\mathscr{HS}_n$ are topologically essentially the same (Theorem 5.3). For convenience, the theorems will be stated and proofs given for the case $n = 8$ only. Similar theorems hold for the cases $n = 2, 4$, and their proofs follow the same line and are simpler.

Theorem 5.1. *For $n = 8$, let us identify the space Q_8 of Cayley numbers with R^8 by regarding the Cayley number*

$$X \equiv (x_1 + x_2 i + x_3 j + x_4 k,\ x_5 + x_6 i + x_7 j + x_8 k)$$

as the point in R^8 with rectangular coordinates $(x_1, ..., x_8)$, and the space $Q_8 \times Q_8$ of ordered pairs of Cayley numbers with $R^8 \times R^8 = R^{16}$ by regarding the ordered pair

$$(X, Y) \equiv ((x_1 + x_2 i + x_3 j + x_4 k,\ x_5 + x_6 i + x_7 j + x_8 k),$$
$$(x_9 + x_{10} i + x_{11} j + x_{12} k,\ x_{13} + x_{14} i + x_{15} j + x_{16} k))$$

as the point in R^{16} with rectangular coordinates $(x_1, ..., x_8; x_9, ..., x_{16})$. Then, written in terms of $x = [x_1 \dots x_8]$ and $y = [x_9 \dots x_{16}]$,

(i) *the equation* $X = 0$ *becomes* $x = 0$;

(ii) *the equation* $Y = CX$ *becomes* $y = xB(\lambda)$, *where* $B(\lambda)$ *is the* 8×8 *matrix in Theorem 1.6 (iii) and* $\lambda = (\lambda_0, \lambda_1, ..., \lambda_7)$ *is related to* C *by*

$$C = (\lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k,\ \lambda_4 + \lambda_5 i + \lambda_6 j + \lambda_7 k). \tag{5.1}$$

Proof. Since (i) is obvious, we shall prove (ii) only. Let $X = (p, q)$, $Y = (r, s)$ and $C = (a, b)$. Then the equation $Y = CX$ is

$$(r, s) = (a, b)(p, q) = (ap - q^* b,\ qa + bp^*),$$

i.e.,

$$\begin{aligned}
&(x_9 + x_{10} i + x_{11} j + x_{12} k,\ x_{13} + x_{14} i + x_{15} j + x_{16} k) \\
&\quad = (\lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k,\ \lambda_4 + \lambda_5 i + \lambda_6 j + \lambda_7 k) \\
&\qquad \times (x_1 + x_2 i + x_3 j + x_4 k,\ x_5 + x_6 i + x_7 j + x_8 k) \\
&\quad = \big((\lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k)(x_1 + x_2 i + x_3 j + x_4 k) \\
&\qquad - (x_5 - x_6 i - x_7 j - x_8 k)(\lambda_4 + \lambda_5 i + \lambda_6 j + \lambda_7 k), \\
&\qquad (x_5 + x_6 i + x_7 j + x_8 k)(\lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k) \\
&\qquad + (\lambda_4 + \lambda_5 i + \lambda_6 j + \lambda_7 k)(x_1 - x_2 i - x_3 j - x_4 k)\big) \\
&\quad = \big((\lambda_0 x_1 - \lambda_1 x_2 - \lambda_2 x_3 - \lambda_3 x_4) - (x_5 \lambda_4 + x_6 \lambda_5 + x_7 \lambda_6 + x_8 \lambda_7) \\
&\qquad + (\lambda_0 x_2 + \lambda_1 x_1 + \lambda_2 x_4 - \lambda_3 x_3) i - (x_5 \lambda_5 - x_6 \lambda_4 - x_7 \lambda_7 + x_8 \lambda_6) i \\
&\qquad + (\lambda_0 x_3 - \lambda_1 x_4 + \lambda_2 x_1 + \lambda_3 x_2) j - (x_5 \lambda_6 + x_6 \lambda_7 - x_7 \lambda_4 - x_8 \lambda_5) j \\
&\qquad + (\lambda_0 x_4 + \lambda_1 x_3 - \lambda_2 x_2 + \lambda_3 x_1) k - (x_5 \lambda_7 - x_6 \lambda_6 + x_7 \lambda_5 - x_8 \lambda_4) k, \\
&\qquad (x_5 \lambda_0 - x_6 \lambda_1 - x_7 \lambda_2 - x_8 \lambda_3) + (\lambda_4 x_1 + \lambda_5 x_2 + \lambda_6 x_3 + \lambda_7 x_4) \\
&\qquad + (x_5 \lambda_1 + x_6 \lambda_0 + x_7 \lambda_3 - x_8 \lambda_2) i + (-\lambda_4 x_2 + \lambda_5 x_1 - \lambda_6 x_4 + \lambda_7 x_3) i \\
&\qquad + (x_5 \lambda_2 - x_6 \lambda_3 + x_7 \lambda_0 + x_8 \lambda_1) j + (-\lambda_4 x_3 + \lambda_5 x_4 + \lambda_6 x_1 - \lambda_7 x_2) j \\
&\qquad + (x_5 \lambda_3 + x_6 \lambda_2 - x_7 \lambda_1 + x_8 \lambda_0) k + (-\lambda_4 x_4 - \lambda_5 x_3 + \lambda_6 x_2 + \lambda_7 x_1) k\big),
\end{aligned}$$

which is easily seen to be equivalent to

$$[x_9 \dots x_{16}] = [x_1 \dots x_8] \begin{bmatrix}
\lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 \\
-\lambda_1 & \lambda_0 & \lambda_3 & -\lambda_2 & \lambda_5 & -\lambda_4 & -\lambda_7 & \lambda_6 \\
-\lambda_2 & -\lambda_3 & \lambda_0 & \lambda_1 & \lambda_6 & \lambda_7 & -\lambda_4 & -\lambda_5 \\
-\lambda_3 & \lambda_2 & -\lambda_1 & \lambda_0 & \lambda_7 & -\lambda_6 & \lambda_5 & -\lambda_4 \\
-\lambda_4 & -\lambda_5 & -\lambda_6 & -\lambda_7 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\
-\lambda_5 & \lambda_4 & -\lambda_7 & \lambda_6 & -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\
-\lambda_6 & \lambda_7 & \lambda_4 & -\lambda_5 & -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\
-\lambda_7 & -\lambda_6 & \lambda_5 & \lambda_4 & -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0
\end{bmatrix},$$

i.e. to $y = xB(\lambda)$.

An immediate consequence of Theorem 5.1 and Theorem 1.6 (iii) is the following

THEOREM 5.2. *The Hopf fibering* $S^{15} \to S^8$ *and our maximal set of mutually isoclinic* 8*-planes in* R^{16} *are equivalent concepts. More precisely, under the identification of* $Q_8 \times Q_8$ *with* R^{16} *described in Theorem 5.1, the set of* Q_8*-lines* $\{X = 0, Y = CX\}$ *in* $Q_8 \times Q_8$ *corresponds to the maximal set* Φ_8 *of mutually isoclinic* 8*-planes* $\{x = 0, y = xB(\lambda)\}$ *in* R^{16}.

(In Appendix 2, we shall prove, by working directly with Cayley numbers, that the Q_8-lines, regarded as 8-planes in R^{16}, are mutually isoclinic 8-planes.)

We are now ready to prove our main

THEOREM 5.3. *The representative coordinate bundles constructed in* § 4 *and* § 3 *for the sphere bundles* $\mathscr{HS}_8$ *and* $\mathscr{I}_8$ *are topologically essentially the same, with only the group* $SO(8)$ *in* $\mathscr{I}_8$ *replacing the group* $O(8)$ *in* $\mathscr{HS}_8$.

Proof. We first identify the bundle space, fiber and base space in $\mathscr{HS}_8$ with those in $\mathscr{I}_8$, and then show that, under this identification, the projection, coordinate functions and coordinate transformations in $\mathscr{HS}_8$ correspond to those in $\mathscr{I}_8$.

(a) The bundle spaces and the fibers.

The bundle space in $\mathscr{HS}_8$ is the unit sphere $S^{15}: |X|^2 + |Y|^2 = 1$ in $Q_8 \times Q_8$, and that in $\mathscr{I}_8$ is the unit sphere $S^{15}: xx^T + yy^T = 1$ in R^{16}. The fiber in $\mathscr{HS}_8$ is the unit sphere $S^7: |X| = 1$ in Q_8, and that in $\mathscr{I}_8$ is the unit sphere $S^7: tt^T = 1$ in R^8. Let us identify these two S^{15}'s and two S^7's by identifying Q_8 with R^8 and $Q_8 \times Q_8$ with $R^{16} = R_8 \times R_8$ as in Theorem 5.1.

(b) The base spaces.

By definition, the base space in $\mathscr{HS}_8$ is $S^8 \equiv Q_8 \cup \infty$ covered by the open sets

$$\{Q_8, S^8 \backslash O = (Q_8 \backslash O) \cup \infty\},$$

with the Cayley numbers and ∞ serving as coordinates. On the other hand, the base space in $\mathscr{I}_8$ is Φ_8 covered by the open sets

$$\{\Phi_8 \backslash \mathbf{O}^{\perp}, \Phi_8 \backslash \mathbf{O}\}$$

such that an 8-plane $y = xB(\lambda)$ in $\Phi_8\backslash\mathbf{O}^\perp$ has the coordinate λ and an 8-plane $x = yB(\mu)^T$ in $\Phi_8\backslash\mathbf{O}$ has the coordinate μ.

Now $Q_8 \cup \infty$ can be regarded as the image of the unit sphere S^8 in R^9 under the stereographic projection from the north pole of S^8 onto the equator 8-plane, and (by Theorem 2.3) there is a homeomorphism from Φ_8 to S^8 which sends the 8-plane $y = xB(\lambda)$ in Φ_8 to the point of S^8 whose stereographic projection is the point λ on the equator 8-plane. Therefore, we can identify $Q_8 \cup \infty$ with Φ_8 by means of a homeomorphism defined as follows.

Let $j_1 : Q_8 \to \Phi_8\backslash\mathbf{O}^\perp$ be the map which sends the point

$$C = (\lambda_0+\lambda_1 i+\lambda_2 j+\lambda_3 k,\ \lambda_4+\lambda_5 i+\lambda_6 j+\lambda_7 k)$$

in Q_8 to the 8-plane $y = xB(\lambda)$ in $\Phi_8\backslash\mathbf{O}^\perp$ with coordinate

$$\lambda = (\lambda_0, \lambda_1, ..., \lambda_7);$$

and $j_2 : S^8\backslash O = (Q_8\backslash O) \cup \infty \to \Phi_8\backslash\mathbf{O}$ the map which sends the point

$$C = (\mu_0+\mu_1 i+\mu_2 j+\mu_3 k,\ \mu_4+\mu_5 i+\mu_6 j+\mu_7 k)/N(\mu)$$

in $Q_8\backslash O \subset S^8\backslash O$ to the 8-plane $x = yB(\mu)^T$ in $\Phi_8\backslash\mathbf{O}$ with coordinate

$$\mu = (\mu_0, \mu_1, ..., \mu_7),$$

and the point $\infty \in S^8\backslash O$ to the 8-plane $\mathbf{O}^\perp : x = 0$ in $\Phi_8\backslash\mathbf{O}$ with coordinate $\mu = 0$. Then it follows easily from Theorem 2.3 and its proof that the map $j_1 \cup j_2$ is a homeomorphism from the base space $S^8 = Q_8 \cup \infty$ in $\mathscr{HS}_8$ to the base space Φ_8 in $\mathscr{I}_8$.

Let us identify these two base spaces by means of the homeomorphism $j_1 \cup j_2$.

(c) The projections.

We now prove that, under the identification defined in (a) and (b) above, the projection p in $\mathscr{HS}_8$ coincides with the projection π in $\mathscr{I}_8$. Suppose that P is a point of S^{15} lying on the Q_8-line $Y = CX$. Then $p(P) = C \in Q_8 \subset S^8$. Now by Theorem 5.1, this point P lies on the 8-plane $y = xB(\lambda)$ in R^{16}, where $\lambda = (\lambda_0, \lambda_1, ..., \lambda_7)$ is related to C by (5.1). Therefore, under the identification of S^8 and Φ_8 defined in (b), $p(P) = \pi(P)$. Suppose now that P is a point of S^{15} lying on the Q_8-line $X = 0$. Then $p(P) = \infty \in S^8$. By Theorem 5.1, this point P lies on the 8-plane $\mathbf{O}^\perp : x = 0$ in R^{16}. Therefore, $\pi(P)$ is the 8-plane $\mathbf{O}^\perp$ in Φ_8. Since the point $\infty \in S^8$ corresponds

to the 8-plane $\mathbf{O}^{\perp}$ in Φ_8 under the identification defined in (b), $p(P) = \pi(P)$. Hence our proof that p and π coincide is complete.

(d) The coordinate functions.

Consider first the coordinate functions

$$\psi_1 : Q_8 \times S^7 \to p^{-1}(Q_8) \quad \text{and} \quad \phi_1 : (\Phi_8 \backslash \mathbf{O}^{\perp}) \times S^7 \to \pi^{-1}(\Phi_8 \backslash \mathbf{O}^{\perp})$$

in $\mathscr{HS}_8$ and $\mathscr{I}_8$, given by (4.2) and (3.1) respectively. Suppose that under the identification defined in (b) and (a), the element $(C, D) \in Q_8 \times S^7$ corresponds to the element $(\lambda, t) \in (\Phi_8 \backslash \mathbf{O}) \times S^7$. Then C and λ are related by

$$C = (\lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k,\ \lambda_4 + \lambda_5 i + \lambda_6 j + \lambda_7 k), \tag{5.1}$$

and D and t by

$$D = (t_1 + t_2 i + t_3 j + t_4 k,\ t_5 + t_6 i + t_7 j + t_8 k). \tag{5.2}$$

Now since $D \in S^7 \subset Q_8$ and $t \in S^7 \subset R^8$, we have $|D| = 1$ and $tt^T = 1$, and, by Theorem 5.1, the product CD corresponds to $tB(\lambda)$. Therefore,

$$|C|^2 = |C|^2 |D|^2 = |CD|^2 = tB(\lambda)\left(tB(\lambda)\right)^T = tB(\lambda)B(\lambda)^T t^T = N(\lambda),$$

and

$$\psi_1(C, D) = \frac{(D, CD)}{\sqrt{1 + |C|^2}} \quad \text{corresponds to} \quad \phi_1(\lambda, t) = \frac{(t, tB(\lambda))}{\sqrt{1 + N(\lambda)}}.$$

Next, consider the coordinate functions

$$\psi_2 : (S^8 \backslash O) \times S^7 \to p^{-1}(S^8 \backslash O) \quad \text{and} \quad \phi_2 : (\Phi_8 \backslash \mathbf{O}) \times S^7 \to \pi^{-1}(\Phi_8 \backslash \mathbf{O})$$

in $\mathscr{HS}_8$ and $\mathscr{I}_8$, given by (4.3) and (3.2) respectively. Suppose that under the identification defined in (b) and (a), the element $(C, D) \in (S^8 \backslash O) \times S^7$ corresponds to the element $(\mu, t') \in (\Phi_8 \backslash \mathbf{O}) \times S^7$. Then, C and μ are related by

$$C = (\mu_0 + \mu_1 i + \mu_2 j + \mu_3 k,\ \mu_4 + \mu_5 i + \mu_6 j + \mu_7 k)/N(\mu), \tag{5.3}$$

and D and t' by

$$D = (t'_1 + t'_2 i + t'_3 j + t'_4 k,\ t'_5 + t'_6 i + t'_7 j + t'_8 k). \tag{5.4}$$

Since (5.3) implies that $|C|^2 = N(\mu)^{-1}$, (5.3) is equivalent to

$$C^{-1} = C^*/|C|^2 = (\mu_0 - \mu_1 i - \mu_2 j - \mu_3 k,\ -\mu_4 - \mu_5 i - \mu_6 j - \mu_7 k). \tag{5.3$'$}$$

Therefore, by Theorem 5.1, $C^{-1}D$ corresponds to

$$t'(\mu_0 - \mu_1 B_1 - \ldots - \mu_7 B_7) = t' B(\mu)^T.$$

Hence, it follows from the above that

$$\psi_2(C, D) = \frac{(C^{-1}D, D)}{\sqrt{1 + 1/|C|^2}} \quad \text{corresponds to } \phi_2(\mu, t') = \frac{(t'B(\mu)^T, t')}{\sqrt{1 + N(\mu)}}.$$

To complete the proof that ψ_2 and ϕ_2 correspond, we need only observe that under the identification defined in (a) and (b), the point $\infty \in S^8\backslash O$ corresponds to the 8-plane $\mathbf{O}^\perp : x = 0$ in $\Phi_8\backslash\mathbf{O}$, and the point

$$\psi_2(\infty, D) = (O, D) \in p^{-1}(S^8\backslash O)$$

of S^{15} coincides with the point

$$\phi_2(O, t') = (O, t') \in \pi^{-1}(\Phi_8\backslash\mathbf{O}).$$

(e) The coordinate transformations and the bundle groups.

Suppose that in $\mathscr{HS}_8$, C is any point in the intersection $Q_8 \cap (S^8\backslash O) = Q_8\backslash O$ of the two coordinate neighborhoods in the base space S^8, and $D \in Q_8$ with $|D| = 1$ is a variable point of the fiber S^7. Then the coordinate transformations in the fiber S^7 are $D \to CD/|C|$ (cf. proof of Theorem 4.1). Now let λ be the point in the intersection $(\Phi_8\backslash\mathbf{O}^\perp) \cap (\Phi_8\backslash\mathbf{O}) = \Phi_8\backslash\{\mathbf{O}^\perp, \mathbf{O}\}$ of two coordinate neighborhoods in the base space Φ_8 in $\mathscr{I}_8$ corresponding to the point C under the identification defined in (b) above, and $t \in R^8$ with $tt^T = 1$ the point of the fiber S^7 in $\mathscr{I}_8$ corresponding to the point D under the identification defined in (a). Then C and λ are related by (5.1), and D and t are related by (5.2). Since (5.1) implies that $|C|^2 = N(\lambda)$ and since by Theorem 5.1. CD corresponds to $tB(\lambda)$, the coordinate transformations $D \to CD/|C|$ in $\mathscr{HS}_8$ correspond to the coordinate transformations $t \to tB(\lambda)/N(\lambda)^{1/2}$ in $\mathscr{I}_8$.

Since the bundle group of a coordinate bundle may be taken as the group generated by the coordinate transformations in the fiber, or any effective transformation group of the fiber containing this group, it follows from the above that the bundle groups in $\mathscr{HS}_8$ and $\mathscr{I}_8$ are the same. Now, the bundle group in $\mathscr{HS}_8$ as originally given by N. Steenrod is $O(8)$; whereas, we have shown in § 3 that the bundle group in $\mathscr{I}_8$ is $G_8 = SO(8)$ and, moreover, it cannot be replaced by any smaller subgroup of $SO(8)$.

The proof of Theorem 5.3 is now complete.

Let us now consider the cases $n = 2$ and 4. By using the results similar to those in Theorem 5.1 for the case $n = 8$, we can prove, as in (e) above, that the coordinate transformations $D \to CD/|C|$ in $\mathscr{HS}_4$

correspond to the coordinate transformations $t \to tB(\lambda)/N(\lambda)^{1/2}$ in $\mathscr{I}_4$, where $B(\lambda)$ are the matrices given in (1.7) in Theorem 1.6. By Theorem 2.5, the elements $B(\lambda)/N(\lambda)^{1/2}$ of $SO(4)$ form a subgroup isomorphic with S^3. Therefore, the bundle group $O(4)$ in $\mathscr{HS}_4$ can be replaced by S^3. Similarly, the bundle group $O(2)$ in $\mathscr{HS}_2$ can be replaced by S^1. With these observations, we can now prove the following theorem by proceeding as in the proof of Theorem 5.3.

THEOREM 5.4. *The representative coordinate bundles constructed in* § 4 *for the sphere bundles* $\mathscr{HS}_2$ *and* $\mathscr{HS}_4$, *with bundle groups* S^1 *and* S^3 *respectively, are topologically the same as the representative coordinate bundles constructed in* § 3 *for the sphere bundles* $\mathscr{I}_2$ *and* $\mathscr{I}_4$, *respectively.*

Finally, we remark that representative coordinate bundles of the bundles $\mathscr{SL}_n$ in Theorem 4.2 are topologically essentially the same as the representative coordinate bundles of the bundles $\mathscr{IL}_n$ in Theorem 3.2.

APPENDIX 1. THE CAYLEY NUMBERS

The Cayley numbers, denoted by X, Y, Z, W, etc. are ordered pairs (q_1, q_2) of quaternions subject to the rules and having the properties listed below. The set of all Cayley numbers, therefore, forms a (non-commutative and non-associative) real division algebra. No proof of the properties will be given as they can all be checked by direct computations.

(1) The *addition* is defined by

$$(q_1, q_2) + (q'_1, q'_2) = (q_1 + q'_1, q_2 + q'_2).$$

The *zero* is $O = (O, O)$.

(2) The *multiplication* is defined by

$$(q_1, q_2)(q'_1, q'_2) = (q_1 q'_1 - q'^*_2 q_2, q'_2 q_1 + q_2 q'^*_1),$$

where q'^*_1, q'^*_2 are respectively the conjugates of (the quaternions) q'_1, q'_2. The (two-sided) *unit* is $1 \equiv (1, 0)$.

(3) Multiplication is

(i) distributive with respect to addition, i.e.,

$$(X+Y)W = XW + YW, \quad W(X+Y) = WX + WY;$$