

## D) Decomposition of primes

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If  $d \equiv 1 \pmod{4}$  then  $\left\{1, \frac{1 + \sqrt{d}}{2}\right\}$  is an integral basis of  $A$ .

### C) DISCRIMINANT

Let  $\{\alpha_1, \alpha_2\}$  be an integral basis. Then

$$D = D_K = \det \begin{pmatrix} \text{Tr}(\alpha_1^2) & \text{Tr}(\alpha_1\alpha_2) \\ \text{Tr}(\alpha_1\alpha_2) & \text{Tr}(\alpha_2^2) \end{pmatrix}$$

is independent of the choice of the integral basis. It is called the discriminant of  $K$ . It is a non-zero integer.

If  $d \equiv 2$  or  $3 \pmod{4}$  then

$$D = \det \begin{pmatrix} \text{Tr}(1) & \text{Tr}(\sqrt{d}) \\ \text{Tr}(\sqrt{d}) & \text{Tr}(d) \end{pmatrix} = \det \begin{pmatrix} 2 & 0 \\ 0 & 2d \end{pmatrix} \quad \text{so } D = 4d.$$

If  $d \equiv 1 \pmod{4}$  then

$$D = \det \begin{pmatrix} \text{Tr}(1) & \text{Tr}\left(\frac{1+\sqrt{d}}{2}\right) \\ \text{Tr}\left(\frac{1+\sqrt{d}}{2}\right) & \text{Tr}\left(\frac{1+\sqrt{d}}{2}\right)^2 \end{pmatrix} = \det \begin{pmatrix} 2 & 1 \\ 1 & \frac{1+d}{2} \end{pmatrix} \quad \text{so } D = d.$$

Every discriminant is  $D \equiv 0$  or  $1 \pmod{4}$ .

In terms of the discriminant,

$$A = \left\{ \frac{a + b\sqrt{D}}{2} \mid a, b \in \mathbf{Z}, \quad a^2 \equiv Db^2 \pmod{4} \right\}.$$

### D) DECOMPOSITION OF PRIMES

Let  $K = \mathbf{Q}(\sqrt{d})$ , where  $d$  is a square-free integer, let  $A$  be the ring of integers of  $K$ .

The ideal  $P \neq 0$  of  $A$  is a prime ideal if the residue ring  $A/P$  has no zero-divisors.

If  $P$  is a prime ideal there exists a unique prime number  $p$  such that  $P \cap \mathbf{Z} = \mathbf{Z}p$ , or equivalently,  $P \supseteq Ap$ .

If  $I, J$  are non-zero ideals of  $A$ , it is said that  $I$  divides  $J$  when there exists an ideal  $I_1$  of  $A$  such that  $I \cdot I_1 = J$ .

The prime ideal  $P$  containing the prime number  $p$  divides the ideal  $Ap$ .

If  $I$  is a non-zero ideal of  $A$  then the residue ring  $A/I$  is finite. The norm of  $I$  is  $N(I) = \#(A/I)$ .

Properties of the norm:

If  $I, J$  are non-zero ideals, then  $N(I \cdot J) = N(I) N(J)$ .

If  $I$  divides  $J$  then  $N(I)$  divides  $N(J)$ .

If  $\alpha \in A$ ,  $\alpha \neq 0$ , then  $N(A\alpha) = |N(\alpha)|$  (absolute value of the norm of  $\alpha$ ). In particular, if  $a \in \mathbf{Z}$  then  $N(Aa) = a^2$ .

If the prime ideal  $P$  divides  $Ap$  then  $N(P)$  is equal to  $p$  or to  $p^2$ .

Every ideal  $I \neq 0$  is, in unique way, the product of powers of prime ideals:

$$I = \prod_{i=1}^n P_i^{e_i}.$$

If  $I, J$  are non-zero ideals, if  $I \supseteq J$  then  $I$  divides  $J$ .

Every ideal  $I \neq 0$  may be generated by two elements, of which one may be chosen in  $\mathbf{Z}$ ; if  $I \cap \mathbf{Z} = \mathbf{Z}n$  then  $I = An + A\alpha$  for some  $\alpha \in A$ . In this case, the following notation is used:  $I = (n, \alpha)$ .

Consider now the special case where  $p$  is a prime number. Then  $Ap$  is of one of the following types:

$$\left\{ \begin{array}{l} Ap = P^2, \quad \text{where } P \text{ is a prime ideal: } p \text{ is ramified in } K. \\ Ap = P, \quad \text{where } P \text{ is a prime ideal: } p \text{ is inert in } K. \\ Ap = P_1 P_2, \text{ where } P_1, P_2 \text{ are distinct prime ideals: } p \text{ is decomposed or splits in } K. \end{array} \right.$$

Note also that if  $Ap = I \cdot J$ , where  $I, J$  are any ideals (different from  $A$ ), not necessarily distinct, then  $I, J$  must in fact be prime ideals.

I shall now indicate when a prime number  $p$  is ramified, inert or decomposed, and also give generators of the prime ideals of  $A$ . There are two cases:  $p \neq 2$ ,  $p = 2$ .

Denote by  $\left(\frac{d}{p}\right)$  the Legendre symbol, so

$$\left\{ \begin{array}{l} \left(\frac{d}{p}\right) = 0 \quad \text{when } p \text{ divides } d, \\ \left(\frac{d}{p}\right) = +1 \quad \text{when } d \text{ is a square modulo } p, \\ \left(\frac{d}{p}\right) = -1 \quad \text{when } d \text{ is not a square modulo } p. \end{array} \right.$$

Let  $p \neq 2$ .

- 1) If  $p$  divides  $d$  then  $Ap = (p, \sqrt{d})^2$ .
- 2) If  $p$  does not divide  $d$  and there does not exist  $a \in \mathbf{Z}$  such that  $d \equiv a^2 \pmod{p}$  then  $Ap$  is a prime ideal.
- 3) If  $p$  does not divide  $d$  and there exists  $a \in \mathbf{Z}$  such that  $d \equiv a^2 \pmod{p}$  then  $Ap = (p, a + \sqrt{d})(p, a - \sqrt{d})$ .

Hence

- 1)  $p$  is ramified if and only if  $\left(\frac{d}{p}\right) = 0$ .
- 2)  $p$  is inert if and only if  $\left(\frac{d}{p}\right) = -1$ .
- 3)  $p$  is decomposed if and only if  $\left(\frac{d}{p}\right) = +1$ .

*Proof.* The proof is divided into several parts.

- a) If  $\left(\frac{d}{p}\right) = -1$  then  $Ap$  is a prime ideal.

Otherwise  $Ap = P \cdot P'$  or  $P^2$ , with  $P \cap \mathbf{Z} = \mathbf{Z}p$ . Let  $\alpha \in A$  be such that  $P = (p, \alpha) \supseteq A\alpha$  so  $P \mid A\alpha$ , hence  $p$  divides  $N(P)$ , which divides  $N(A\alpha) = |N(\alpha)|$ . If  $p \mid \alpha$  then  $\frac{\alpha}{p} \in A$  and  $P = Ap \cdot \left(1, \frac{\alpha}{p}\right) = Ap$ , which is absurd. So  $p \nmid \alpha$ . Then,

$$\begin{aligned} \begin{cases} d \equiv 2 \text{ or } 3 \pmod{4} \\ d \equiv 1 \pmod{4} \end{cases} &\Rightarrow \begin{cases} \alpha = a + b\sqrt{d}, \text{ with } a, b \in \mathbf{Z} \\ \alpha = \frac{a + b\sqrt{d}}{2}, \text{ with } a, b \in \mathbf{Z}, a \equiv b \pmod{2} \end{cases} \\ &\Rightarrow \begin{cases} N(\alpha) = a^2 - db^2 \\ N(\alpha) = \frac{a^2 - db^2}{4} \end{cases} \Rightarrow p \text{ divides } a^2 - db^2, \end{aligned}$$

hence  $a^2 \equiv db^2 \pmod{p}$  and so  $p \nmid b$  (otherwise  $p \mid a$ , hence  $p \mid \alpha$ , which is absurd).

Let  $b'$  be such that  $bb' \equiv 1 \pmod{p}$ , so  $(ab')^2 \equiv d \pmod{p}$ , therefore either  $p \mid d$  or  $\left(\frac{d}{p}\right) = +1$ , which is a contradiction.

b) If  $\left(\frac{d}{p}\right) = 0$  then  $Ap = (p, \sqrt{d})^2$ .

Indeed, let  $P = (p, \sqrt{d})$ , so  $P^2 = (p^2, p\sqrt{d}, d) = Ap\left(p, \sqrt{d}, \frac{d}{p}\right)$  since  $\frac{d}{p} \in \mathbf{Z}$ . But  $d$  is square-free, so  $\gcd\left(p, \frac{d}{p}\right) = 1$ , hence  $P^2 = Ap$  and this implies that  $P$  is a prime ideal.

c) If  $\left(\frac{d}{p}\right) = -1$  then  $Ap = (p, a + \sqrt{d})(p, a - \sqrt{d})$ , where  $1 \leq a \leq p - 1$  and  $a^2 \equiv d \pmod{p}$ .

Indeed,

$$\begin{aligned} (p, a + \sqrt{d})(p, a - \sqrt{d}) &= (p^2, pa + p\sqrt{d}, pa - p\sqrt{d}, a^2 - d) \\ &= Ap\left(p, a + \sqrt{d}, a - \sqrt{d}, \frac{a^2 - d}{p}\right) = Ap\left(p, a + \sqrt{d}, a - \sqrt{d}, 2a, \frac{a^2 - d}{p}\right) = Ap, \end{aligned}$$

because  $\gcd(p, 2a) = 1$ . If one of the ideals  $(p, a + \sqrt{d})$ ,  $(p, a - \sqrt{d})$  is equal to  $A$ , so is the other which is not possible.

So  $(p, a + \sqrt{d})$ ,  $(p, a - \sqrt{d})$  are prime ideals. They are distinct: if  $(p, a + \sqrt{d}) = (p, a - \sqrt{d})$  then they are equal to their sum

$$(p, a + \sqrt{d}, a - \sqrt{d}) = (p, a + \sqrt{d}, a - \sqrt{d}, 2a) = A,$$

which is an absurd.

Finally, these three cases are exclusive and exhaustive, so the converse assertions are also true.  $\square$

*Note.* If  $d \equiv 1 \pmod{4}$  and  $d \equiv a^2 \pmod{p}$  then

$$(p, a + \sqrt{d}) = (p, l(a - 1) + \omega),$$

where  $\omega = \frac{1 + \sqrt{d}}{2}$  and  $2l \equiv 1 \pmod{p}$ . Hence, if  $\left(\frac{d}{p}\right) \neq -1$  there exists  $b \in \mathbf{Z}$ ,  $0 \leq b \leq p - 1$ , such that  $p$  divides  $N(b + \omega)$  and moreover if  $b = p - 1$  then  $d \equiv 1 \pmod{p}$ .

Indeed,  $a + \sqrt{d} = a - 1 + 2\omega$ . If  $2l \equiv 1 \pmod{p}$  then

$$(p, a + \sqrt{d}) = (p, (a - 1) + 2\omega) = (p, l(a - 1) + \omega).$$

If  $\left(\frac{d}{p}\right) \neq -1$  then there exists a prime ideal  $P$  dividing  $Ap$ , where

$$P = (p, a + \sqrt{d}), 0 \leq a \leq p-1.$$

So  $P = (p, b + \omega)$  with  $0 \leq b \leq p-1$ ,  $b \equiv l(a-1) \pmod{p}$ .

Since  $P \supseteq A(b + \omega)$  then  $p$  divides  $N(P)$ , which divides  $N(b + \omega)$ . Finally, if  $p$  divides  $N(p-1 + \omega) = N\left(\frac{2p-1+\sqrt{d}}{2}\right) = \frac{(2p-1)^2 - d}{4}$  then  $p$  divides  $\frac{1-d}{4}$  so  $d \equiv 1 \pmod{p}$ .

Let  $p = 2$ .

If  $d \equiv 2 \pmod{4}$  then  $A2 = (2, \sqrt{d})^2$ .

If  $d \equiv 3 \pmod{4}$  then  $A2 = (2, 1 + \sqrt{d})^2$ .

If  $d \equiv 1 \pmod{8}$  then  $A2 = (2, \omega)(2, \omega')$ .

If  $d \equiv 5 \pmod{8}$  then  $A2$  is a prime ideal.

Hence

- 1) 2 is ramified if and only if  $d \equiv 2$  or  $3 \pmod{4}$ .
- 2) 2 is inert if and only if  $d \equiv 5 \pmod{8}$ .
- 3) 2 is decomposed if and only if  $d \equiv 1 \pmod{8}$ .

*Proof.* The proof is divided into several parts.

a) If  $d \equiv 5 \pmod{8}$  then  $A2$  is a prime ideal.

Otherwise,  $A2 = P \cdot P'$  or  $P^2$ , with  $P \cap \mathbf{Z} = \mathbf{Z}2$ . Then there exists  $\alpha \in A$  such that  $P = (2, \alpha) \supseteq A\alpha$ , so  $P$  divides  $A\alpha$  and 2 divides  $N(P)$ , which divides  $N(\alpha)$ .

If  $2 \mid \alpha$  then  $P = A2\left(l, \frac{\alpha}{2}\right) = A2$ , which is absurd. Thus

$$2 \nmid \alpha = \frac{a + b\sqrt{d}}{2}, \quad \text{with} \quad a \equiv b \pmod{2}, \quad \text{so} \quad N(\alpha) = \frac{a^2 - db^2}{4}.$$

From  $2 \mid N(\alpha)$  then 8 divides  $a^2 - db^2 \equiv a^2 - 5b^2 \equiv a^2 + 3b^2 \pmod{8}$ .

If  $a, b$  are odd then  $a^2 \equiv b^2 \equiv 1 \pmod{8}$ , so  $a^2 + 3b^2 \equiv 4 \pmod{8}$ , which is absurd. So  $a, b$  are even,  $a = 2a'$ ,  $b = 2b'$ , and  $\alpha = a' + b'\sqrt{d}$ , 2 divides  $N(\alpha) = a'^2 - db'^2$ .

Since  $d$  is odd, then  $a', b'$  are both even or both odd.

If  $a', b'$  are even then 2 divides  $\alpha$ , which is absurd.

If  $a', b'$  are odd then  $\alpha = a' + b'\sqrt{d} = (\text{multiple of } 2) + 1 + \sqrt{d} = (\text{multiple of } 2) + 2\omega = (\text{multiple of } 2)$ , which is absurd.

b) If  $d \equiv 1 \pmod{8}$  then  $A2 = (2, \omega)(2, \omega')$ .

Indeed,

$$(2, \omega)(2, \omega') = \left(4, 2\omega, 2\omega', \frac{1-d}{4}\right) = A2 \left(2, \omega, \omega', \frac{1-d}{8}\right) = A2,$$

because  $\omega + \omega' = 1$ .

Also  $(2, \omega) \neq (2, \omega')$ , otherwise these ideals are equal to their sum  $(2, \omega, \omega') = A$ , because  $\omega + \omega' = 1$ .

c) If  $d \equiv 2$  or  $3 \pmod{4}$  then  $A2 = (2, \sqrt{d})^2$ , respectively  $(2, 1 + \sqrt{d})^2$ . First let  $d = 4e + 2$  then

$$(2, \sqrt{d})^2 = (4, 2\sqrt{d}, d) = A2(2, \sqrt{d}, 2e+1) = A2,$$

so  $(2, \sqrt{d})$  is a prime ideal.

Now, let  $d = 4e + 3$ , then

$$\begin{aligned} (2, 1 + \sqrt{d})^2 &= (4, 2 + 2\sqrt{d}, 1 + d + 2\sqrt{d}) = (4, 2 + 2\sqrt{d}, 4(e+1) + 2\sqrt{d}) \\ &= A2(2, 1 + \sqrt{d}, 2(e+1) + \sqrt{d}) = A2(2, 2e+1, 1 + \sqrt{d}, 2(e+1) + \sqrt{d}) = A2 \end{aligned}$$

and so  $(2, 1 + \sqrt{d})$  is a prime ideal.

Finally, these three cases are exclusive and exhaustive, so the converse assertions also hold.  $\square$

## E) UNITS

The element  $\alpha \in A$  is a unit if there exists  $\beta \in A$  such that  $\alpha\beta = 1$ . The set  $U$  of units is a group under multiplication. Here is a description of the group of units in the various cases. First let  $d < 0$ .

Let  $d \neq -1, -3$ . Then  $U = \{\pm 1\}$ .

Let  $d = -1$ . Then  $U = \{\pm 1, \pm i\}$ , with  $i = \sqrt{-1}$ .

Let  $d = -3$ . Then  $U = \{\pm 1, \pm \rho, \pm \rho^2\}$ , with  $\rho^3 = 1$ ,  $\rho \neq 1$ , i.e.

$$\rho = \frac{-1 + \sqrt{-3}}{2}.$$

Let  $d > 0$ . Then the group of units is the product  $U = \{\pm 1\} \times C$ , where  $C$  is a multiplicative cyclic group. Thus  $C = \{\varepsilon^n \mid n \in \mathbb{Z}\}$ , where  $\varepsilon$  is the smallest unit such that  $\varepsilon > 1$ .  $\varepsilon$  is called the fundamental unit.