

§2. Topological Buildings

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$y = s_1 \cdots s_k (s_i \in S)$ and x has a reduced decomposition obtained by deleting some subset of the s_i 's occurring in y . (For a very nice account of these related matters, see [14]). If W is finite, W has a unique element w_0 of maximal length, we define the length of W to be $l(w_0)$.

§ 2. TOPOLOGICAL BUILDINGS

A *Tits system* (G, B, N, S) consists of a group G , subgroups B and N , and a set S , which satisfy the following axioms:

- (2.1) $B \cap N$ is normal in N , and S is a set of involutions generating $\bar{W} \equiv N/B \cap N$,
- (2.2) B and N generate G ,
- (2.3) If $s \in S$, $sBs \neq B$,
- (2.4) if $s \in S$, $w \in W$, then $sBw \leq BwB \cup BswB$.

(The use of expressions such as sBw is a standard abuse of notation).

Example. Let G be a reductive algebraic group over an algebraically closed field (e.g., $GL(n, \mathbb{C})$), let B be a Borel subgroup (e.g. upper triangular matrices), and let N be the normalizer of a maximal torus (that lies in B). This data determines a set S of simple reflections generating the Weyl group W (e.g., the usual generators $s_1, \dots, s_{n-1}$ of Σ_n). Then one of the main results in the structure theory of reductive groups is that (G, B, N, S) is a Tits system (see for example [15]).

Throughout this paper we will assume that the set S is finite; its cardinality l is the *rank* of the system.

We next list some of the important properties of a Tits system.

- (2.5) (Bruhat Decomposition) $G = \coprod_{w \in W} BwB$ (disjoint union),
- (2.6) (W, S) is a Coxeter system.

A subgroup P of G is *parabolic* if it contains a conjugate of B . In particular if $I \subseteq S$, the subgroup P_I generated by B and I is parabolic.

- (2.7) (a) The parabolic subgroups containing B are precisely the P_I , $I \subseteq S$. No two of these are conjugate; in particular there are exactly 2^l such subgroups, which form a lattice isomorphic to the lattice of subsets of S .
- (b) $P_I = BW_I B$
- (c) Every parabolic P is self-normalizing: $N_G P = P$.

(2.8) (Bruhat decomposition, general version) $G = \coprod_{w \in W_I \backslash W/W_J} P_I w P_J$ (disjoint union).

The next result, which we will refer to as the *Steinberg Lemma*, is somewhat technical; however it is not hard to prove and is extremely useful. It is a mild generalization of Theorem 15 of [32] and Proposition 3.1 of [19].

(2.9) Let $I \subseteq S$ and suppose w is the unique element of minimal length of wW_I . Suppose $w = w_1 \dots w_k$ where $l(w) = l(w_1) + \dots + l(w_k)$. Then

(a) If Y_i is any subset of Bw_iB such that $Y_i \rightarrow Bw_iB/B$ is bijective (resp. surjective) ($1 \leq i \leq k$), then $Y_1 \times Y_2 \times \dots \times Y_k \rightarrow BwP_I/P_I$ is bijective (resp. surjective).

(b) Suppose $w_i \in S$, $1 \leq i \leq k$ i.e., $w_1 \dots w_k$ is a reduced decomposition of w . Let Z_i , $1 \leq i \leq k$, be any subset containing 1 of P_{w_i} such that $Z_i \rightarrow P_{w_i}/B$ is surjective. Then the image of $Z_1 \times \dots \times Z_k \rightarrow G/P_I$ is $\coprod_{x \leq w} BxP_I/P_I$.

The maps in (a), (b) are the obvious multiplication/projection maps. Part b refers to the Bruhat order on W^I .

(2.10) *Remark.* The Tits system of a reductive algebraic group has several additional features: $B = HU$, where H is a maximal torus and U is a normal unipotent subgroup, U in turn is described in terms of its root subgroups, and there is an "opposite" Borel subgroup B^- such that $B \cap B^- = H$. This additional structure can also be axiomatized in an elegant way, leading to the "refined" Tits system of Kac and Peterson [19]. One then obtains, for example, the *Birkhoff decomposition* $G = \coprod_{w \in W} B^- w B$ as a consequence of the axioms.

We now define a *topological* Tits system to be a Tits system such that G is a topological group, B and N are closed subgroups, and W is discrete (i.e. $N \cap B$ is an open subgroup of N). We will usually also assume (for reasons which will be apparent shortly):

(2.11) *Axiom.* If I is a proper subset of S , W_I is finite.

This axiom is satisfied if W is an irreducible affine Weyl group, or finite. To get any interesting results some further axiom seems necessary. One direction is considered in [11], where the groups in question are algebraic groups over local fields, with the valuation topology. Here, with loop groups in mind, the following axiom seems efficient:

(2.12) *Axiom.* For each $s \in S$ there is a subset A_s of P_s such that (a) $A_s B = P_s$, (b) A_s is compact and contains 1, and (c) $A_s = \overline{A_s \cap BsB}$. This axiom is motivated by Steinberg's approach [32].

(2.13) PROPOSITION. Let (G, B, N, S) be a topological Tits system satisfying (2.12). Then

- (a) $\overline{BwB} = \coprod_{x \leq w} BxB(w \in W)$. More generally if $I \leq S$, and $w \in W^I$, $\overline{BwP_I} = \coprod_{x \leq w} BxP_I$ (here $x \in W^I$),
- (b) B -orbits in G/P_I are locally closed,
- (c) If W satisfies (2.11), parabolic subgroups are closed.

Proof. First we show $P_s = \overline{BsB}$. Since $P_s = A_sB$, with A_s compact and B closed, P_s is closed, so $P_s \geq \overline{BsB}$. But also $B \subset P_s = A_sB \subset \overline{BsB}$, which proves our claim. Part (a) now follows easily from the Steinberg lemma: Let $M_w = \coprod_{x \leq w} BxP_I$, and let $w = s_1 \cdots s_k$ be a reduced decomposition. Then $M_w = A_1 \cdots A_kP_I$ and hence is closed. Next, suppose $x \leq w$; we must show $BxB \leq \overline{BwB}$. It is enough to consider the case when X has a reduced decomposition $x = s_1 \cdots \hat{s}_i \cdots s_k$ (omit s_i). Then

$$BxP_I = A'_1 \cdots A'_{i-1} A'_{i+1} \cdots A'_kP_I \leq A'_1 \cdots A'_{i-1} \bar{A}_i \cdots A'_kP_I \leq \overline{BwP_I}$$

(since $1 \in A_i$), where $A'_i = A_i \cap Bs_iB$. This proves (a). Part (b) is immediate since the complement of BwP_I in its closure is a finite union of sets of the form M_x , hence is closed. Since $P_I = BW_I B$, (c) is also immediate from (a) if W_I is finite. $\square$

From now on we will assume 2.11 and 2.12. The homogeneous spaces G/P_I will be called *flag spaces*. The B -orbits $E_w = BwP_I/P_I$ are *Schubert strata* and the compact subspaces $\overline{E_w}$ are *Schubert subspaces*.

We next consider the *building* $\mathcal{B}_G$ associated to a topological Tits system (G, B, N, S) . (The notation is ambiguous—indeed in the case of loop groups, G will support two natural but totally different Tits system. However the system we have in mind will be clear from the context.) In the discrete case, $\mathcal{B}_G$ is usually defined as the following simplicial complex. The vertices are the maximal (proper) parabolics, and $P_1 \cdots P_k$ span a simplex if $\bigcap_{i=1}^k P_i$ contains a conjugate of B . In general it is convenient to reinterpret this definition as follows: first of all, by definition every parabolic P is conjugate to a unique P_I ; we say that P has type I . Thus the maximal parabolics are the parabolics of type $[s]$, where $[s] = S - \{s\}$. More generally the k -simplices correspond to the parabolics of type I , where $|I| = l - k - 1$. Thus the simplices all have dimension $\leq l - 1$, with the $l - 1$ simplices corresponding to the conjugates of B . Furthermore, in view

of 2.7 (c), the set of parabolics of type I is canonically identified with $G/P_I - xP_I$ corresponding to xP_Ix^{-1} . One can easily check that with this interpretation, a simplex xP_I is a face of a simplex yP_J if and only if $I \supset J$ and $xP_I = yP_I$. In particular, every simplex is a face of some $l - 1$ simplex. Hence, as a set, B_G can be identified with $G/B \times \Delta/\sim$, where Δ is the $l - 1$ simplex with vertex set S , and $(g_1B, X_1) \sim (g_2B, X_2)$ if $X_1 = X = X_2$, $X \in \Delta_I$, and $g_1P_I = g_2P_I$. (Here Δ_I is the face of Δ corresponding to $I \leq S$.) We will therefore *define* the building $\mathcal{B}_G$ associated to the topological Tits system (G, B, N, S) to be $G/B \times \Delta$ modulo this equivalence relation, with the quotient topology.

Remark. Another way of expressing this is as follows: Let C be the category defined by the poset of proper subsets of S (including the empty set). We have a functor from C to topological spaces given by $I \mapsto G/P_I$. Then $\mathcal{B}_G$ is precisely the homotopy colimit of this diagram of spaces, in the sense of [8], p. 327 ff.

(2.14) PROPOSITION. *The equivalence relation on $G/B \times \Delta^{l-1}$ is generated by the relations $(g_1B, X) \sim (g_2B, X)$ if X lies on the wall Δ_s and $g_1P_s = g_2P_s$.*

Proof. In the usual language, (2.14) is the assertion that any two chambers are linked by a "gallery". (See e.g. [11], appendix.) Since the action of G on G/B induces a well defined action on $\mathcal{B}_G$, we are reduced to showing that if $(B, X) \sim (gB, X)$ —i.e. $X \in \Delta_I$ and $g \in P_I$ —then (B, X) and (gB, X) are linked by a sequence of relations of the stated type. But $gB = bwB$ with $w \in W_I$; hence if $w = s_1 \cdots s_k$ is a reduced decomposition, the elements $(B, X), (bs_1B, X), (bs_1s_2B, X), \dots, (bwB, X)$ provide the desired sequence. $\square$

Note that the set Δ is a fundamental domain for the action of G on $\mathcal{B}_G$. On the other hand, it is easy to check that the closed subspace $\mathcal{B}_W$ consisting of the pairs (wB, X) , $w \in W$, is a fundamental domain for the B action. (The point is that if $bw_1P_I = w_2P_I$, then $w_1P_I = w_2P_I$, by the Bruhat decomposition.) This space $\mathcal{B}_W$, which we will call the *foundation* of the building, is a simplicial complex since W is discrete. Since it will turn out that $\mathcal{B}_G$ is in a sense a "thickening" of the foundation, the following well known description of $\mathcal{B}_W$ may be of interest.

(2.15) PROPOSITION. Suppose Φ is an irreducible root system in the Euclidean space V . Then

(a) If W is the affine Weyl group associated to Φ , then $\mathcal{B}_W$ is isomorphic as a simplicial W -complex to V (triangulated by the hyperplanes of Φ).

(b) If W is the Weyl group of Φ , $\mathcal{B}_W$ is isomorphic as simplicial W -complex to the unit sphere of V , triangulated by the Weyl chambers. More precisely, $\mathcal{B}_W$ can be identified with the W orbit of the outer wall of the Cartan simplex.

Proof. For (a), map $W \times \Delta \xrightarrow{\varphi} V$ by identifying Δ with the Cartan simplex in V and using the action map. Then φ is onto (1.1) and furthermore $\varphi(w_1, x) = \varphi(w_2, X_2)$ if and only if $X_1 = X = X_2$, $X \in \Delta_I$, and $w_1 = w_2$ modulo the isotropy group of X . But this isotropy group is precisely W_I (1.2), so φ factors through the desired isomorphism $\mathcal{B}_W \rightarrow V$. The proof of (b) is similar. $\square$

We now come to the main result of this section. Filter G/B by $F_k(G/B) = \coprod_{l(w) \leq k} E_w$. Similarly, $\mathcal{B}_G$ is filtered by $F_k(\mathcal{B}_G) = F_k(G/B) \times \Delta / \sim$.

(2.16) THEOREM. Let (G, B, N, S) be a topological Tits system which either is discrete or satisfies (2.11) and (2.12). Assume also that the inclusions $F_k(B_G) \subset F_{k+1}(B_G)$ are cofibrations. Then

(a) If W is infinite, $\mathcal{B}_G$ is contractible.

(b) If W is finite of length r , $\mathcal{B}_G$ is homotopy equivalent to the $(l-1)$ st suspension $S^{l-1} \wedge (F_r(G/B)/F_{r-1}(G/B))$.

Remark. If G is discrete, $F_k \mathcal{B}_G$ is a subcomplex of the simplicial complex $\mathcal{B}_G$, so the cofibration hypothesis is automatically satisfied. Furthermore if W is finite the smash product in (b) is just a wedge of $|F_r G/B - F_{r-1} G/B|$ $(l-1)$ -spheres. This case is due to Solomon and Tits; cf. [11].

Proof of (2.16). Let X_k denote $F_k \mathcal{B}_G / F_{k-1} \mathcal{B}_G$, and let $X'_k = F_k(G/B) / F_{k-1}(G/B)$. Then we will show

(2.17) If k is less than the length of W , X_k is contractible. If $k = r = \text{length of } W$, X_k is homeomorphic to $(F_r(G/B)/F_{r-1}(G/B) \wedge S^{l-1})$.

If W is infinite, it follows that $F_k \mathcal{B}_G$ is contractible for all k , and hence $\mathcal{B}_G$ is contractible. If W is finite, part (b) of the theorem is also immediate.

To prove 2.17, first consider the quotient map $\pi: F_k(G/B) \times \Delta \rightarrow X_k$. In fact π is merely collapsing a subspace to a point:

(2.18) Let $A_1 = (b_1 w_1 B, X_1)$, $A_2 = (b_2 w_2 B, X_2)$. If $\pi(A_1) = \pi(A_2)$, then either $A_1 = A_2$ or $\pi(A_1) = \pi(A_2) = *$ ($*$ is the basepoint $F_{k-1} B_G$).

For suppose $\pi(A_1) \neq *$, and $X_1 = \dot{\Delta}_I$. Then $l(w_1) = k$ and $w_1 \in W^I$. This forces $X_1 = X_2$ and $w_1 = w_2 \bmod W_I$; hence $w_1 = w_2$ since $l(w_2) \leq k$ by assumption. Then $b_1 w_1 P_I = b_2 w_1 P_I$. But whenever $w \in W^I$, $b_1 w P_I = b_2 w P_I$ implies $b_1 w B = b_2 w B$ (easy exercise).

It now follows that $X_k = \bigvee_{l(w)=k} X_w$, where X_w is the image of $\bar{E}_w \times \Delta$ in X_k , and to prove (2.17) we need only consider a fixed X_w . Let $X'_w = \bar{E}_w / (\bar{E}_w - E_w)$, and let Δ' be the subcomplex of Δ consisting of the walls Δ_s such that $l(ws) < l(w)$. Then (2.18) implies:

$$(2.19) \quad X_w = X'_w \wedge (\Delta / \Delta').$$

For X_w is $\bar{E}_w \times \Delta$ modulo the subspace of points which are equivalent (in $\mathcal{B}_G$) to a point of lower filtration, namely, $\bar{E}_w \times \Delta' \cup \bar{E}_w - E_w \times \Delta$. It remains to identify Δ' . Since $F_0 \mathcal{B}_G = \Delta$ is contractible, we may assume $k \geq 1$; then Δ' is nonempty. If $k < l(W)$, then there is at least one $s \in S$ such that $l(ws) > l(w)$; hence Δ' is not the entire boundary of Δ and Δ / Δ' is contractible. If $k = l(W)$, then w is unique, $\Delta' =$ boundary of Δ , and $\Delta / \Delta' = S^{l-1}$. This completes the proof of (2.17), and of the theorem. $\square$

Remark. Our proof of Theorem 2.16 is an adaptation of the standard (discrete) proof to the topological setting. Much of the proof depends only on the Weyl group W , and indeed shows e.g. for W infinite that the foundation of the building is contractible. In fact the deformation of $F_k(\mathcal{B}_W)$ into $F_{k-1}(\mathcal{B}_W)$ has the property that the isotropy group in B of a point X in $\mathcal{B}_W$ is an increasing function of time, and hence extends uniquely to a B -equivariant deformation of $F_k(B_G)$. In the discrete case this extension is automatically continuous, and shows that Theorem (2.16) holds B -equivariantly. (This was observed, (not for the first time) in [21], and has an interesting application concerning the Steinberg representation of a finite Chevalley group.) However this proof does not work in the topological case; simple counterexamples show that the extension will be discontinuous.

In many cases the Bruhat decomposition of G/P is in fact a CW decomposition. The following axioms are convenient in this regard:

(2.20) *Axiom.* For each $s \in S$, the projection $P_s \rightarrow P_s/B$ has a local section.

(2.21) *Axiom.* For each $s \in S$, P_s/B is homeomorphic to a sphere of positive dimension.

We then have:

(2.22) THEOREM. Let (G, B, N, S) be a topological Tits system satisfying axioms 2.11, 2.20 and 2.21. Let $P \equiv P_I$ be a parabolic subgroup, $I \leq S$, and give G/P the compactly generated topology. Then

(a) Axiom 2.12 is satisfied.

(b) The Bruhat decomposition of G/P is a CW decomposition, and the closure relations on the cells are given by the Bruhat order on W^I .

(c) The building $\mathcal{B}_G$ satisfies the cofibration condition of Theorem 2.16.

Proof. By assumption there are maps $D^{m(s)} \xrightarrow{\varphi_s} P_s/B$ such that $\varphi_s^{-1}(B) = \partial D^{m(s)}$ and $D^{m(s)}/\partial D^{m(s)} \rightarrow P_s/B$ is a homeomorphism. Furthermore φ_s lifts to a map $\tilde{\varphi}_s: D^{m(s)} \rightarrow P_s$ with $1 \in \tilde{\varphi}_s(\partial D^{m(s)})$. Thus, in Axiom (2.12) we may take $A_s = \tilde{\varphi}_s(\mathring{D}^{m(s)})$, proving (a). Since P is closed (2.13c), G/P is a Hausdorff space. If $w \in W^I$ has reduced decomposition $w = s_1 \cdots s_k$, the Steinberg lemma (2.9) shows that the multiplication map $D^{m(s_1)} \times \cdots \times D^{m(s_k)} \rightarrow \bar{E}_w$ (using $\tilde{\varphi}_{s_i}$) is a characteristic map for the cell E_w . The boundary of each cell is a finite union of cells of lower dimension by 2.13a, and G/P has the weak topology by assumption. The closure relations also follow from (2.13). This proves (b). For (c) we observe that $\mathcal{B}_G$ (with the compactly generated topology) is itself a CW-complex, and the filtrations $F_k \mathcal{B}_G$ are subcomplexes: Indeed if we regard $\mathcal{B}_G$ as a quotient space of $\coprod_{I \leq S} (G/P_I \times \Delta_I)$, it is clear that there is one cell for each $I < S$ and $w \in W^I$. $\square$

If G, P_I are as in the above theorem, and $w \in W^I$ has reduced decomposition $w = s_1 \cdots s_k$, let $n(w) = n(s_1) + \cdots + n(s_k)$. Thus $n(w) = \dim E_w$ and so in particular is independent of the choice of reduced decomposition. Now whenever a space has a locally finite cell decomposition, we have a cell series $\sum a_i t^i$, where a_i is the number of cells of dimension i . We then have:

(2.23) COROLLARY. G/P_I admits a CW-decomposition with cell series $\sum_{w \in W^I} t^{n(w)}$. $\square$

Note also:

(2.24) COROLLARY. If W is finite with maximal length element w_0 , $\mathcal{B}_G$ is a sphere of dimension $n(w_0) + l - 1$. $\square$

We conclude this section with two "classical" examples. Let G be a semisimple compact Lie group and consider the Tits system (G, B, N, S) , where B is a Borel subgroup, etc. First we claim that this is a topological Tits system satisfying all four of our axioms. Since W is finite, (2.11) is

trivially satisfied. In (2.12) we can take A_s to be the “little $SU(2)$ ” (or $PSU(2)$) G_s (P_s has Iwasawa decomposition $P_s = G_s B$). In any case there is a commutative diagram

$$\begin{array}{ccc} G_s & \rightarrow & P_s \\ \downarrow & & \downarrow \\ CP^1 = G_s/G_s \cap T & = & P_s/B \end{array}$$

which proves (2.20), (2.21), and hence (2.12) simultaneously. The Bruhat decomposition of $G_{\mathbb{C}}/P_I$, P_I parabolic, is then the classical Schubert cell decomposition of the flag variety $G_{\mathbb{C}}/P_I$. We have $n(s) = 2$ for all s , so $n(w) = 2l(w)$ for all $w \in W^I$. In particular the associated building $\mathcal{B}_{G_{\mathbb{C}}}$ is a sphere of dimension $2l(w_0) + l - 1$ (since $l(w)_0$ is the number of positive roots, this is exactly $\dim G - 1$).

The second example (which is a generalization of the first) involves symmetric spaces G/K and the associated semisimple real Lie group $G_{\mathbb{R}}$ as in § 1. Thus $G_{\mathbb{R}}$ is the fixed group of the involution σ on $G_{\mathbb{C}}$. Now σ need not preserve the Borel subgroup B of $G_{\mathbb{C}}$, but it does preserve the parabolic Q associated to the black nodes of the Satake diagram. We will write $B_{\mathbb{R}}, N_{\mathbb{R}}, W_{\mathbb{R}}, S_{\mathbb{R}}$ for $Q^{\sigma}, N_{K^t m}, W_{G/K}, S_{G/K}$, respectively.

(2.25) THEOREM. $(G_{\mathbb{R}}, B_{\mathbb{R}}, N_{\mathbb{R}}, S_{\mathbb{R}})$ is a topological Tits system satisfying the four axioms. $\square$

A proof that this is a Tits system can be found in [33]. The parabolic subgroups of $G_{\mathbb{R}}$ are related in an obvious way to those of $G_{\mathbb{C}}$: Given $I \subset S_{\mathbb{R}}$, let I' be the corresponding set in S (see § 1). We denote by $\mathcal{O}_I$ the parabolic in $G_{\mathbb{R}}$ generated by $B_{\mathbb{R}}$ and I . Then $\mathcal{O}_I = (P_{I'})^{\sigma}$. ($B_{\mathbb{R}}$ is usually called a “minimal parabolic”, but this terminology conflicts with our use of the term. From the point of view of Tits systems, it is precisely analogous to the Borel subgroup of $G_{\mathbb{C}}$ —although in general it is neither solvable nor connected.) The rest of the theorem is also easily deduced from [33]; the details will be omitted, but see § 5. The main point is that for the minimal parabolics $\mathcal{O}_i$, $\mathcal{O}_i/B_{\mathbb{R}}$ is a sphere of dimension n_i .

As for the building, one can deduce from (2.24) that it is a sphere whose dimension is $\dim G/K - 1$. However it is an interesting fact, that does not seem to appear in the literature, that the building can be canonically identified with the “tangent cut locus” of G/K : first recall (cf. [10], [20]) that if M is a compact Riemannian manifold and p is a fixed point of M , a point

x is a *cut point* (with respect to p) if there is a geodesic from p to x that minimizes arc length up to x but no further. The *cut locus* is the set of cut points. Similarly a vector X in the tangent space T_p is a *tangent cut point* if $\exp_p X$ is a cut point along the geodesic $\exp_p(tX)$. The *tangent cut locus* is the set of all such points in T_p , and is homeomorphic to the unit sphere in T_p . When $M = G/K$ we take $p = 1$.

(2.26) THEOREM. *Let G/K be a simply-connected symmetric space, with G simple. Then the tangent cut locus is precisely the K -orbit in $\mathfrak{m}$ of the outer wall of the Cartan simplex $\Delta_{\mathfrak{m}}$. It is therefore canonically identified with the topological building of the associated real form $G_{\mathbf{R}}$.*

As usual, the assumption G simple is just for convenience. We sketch the proof: the first assertion is a fairly easy consequence of Theorem (1.8), and is left to the reader. Now consider the building $\mathcal{B}_{G_{\mathbf{R}}}$. It is a quotient space of $G_{\mathbf{R}}/B_{\mathbf{R}} \times \Delta_0 = K/C_K t_{\mathfrak{m}} \times \Delta_0$, where Δ_0 is a simplex of dimension $(\text{rank } G/K) - 1$; we take Δ_0 to be the outer wall of $\Delta_{\mathfrak{m}}$. For each $I \leq S_{G/K}$, let Δ_I temporarily denote the corresponding face of Δ_0 ; i.e. $\{X \in \Delta_0 : \alpha_i(X) = 0 \ \forall i \in I\}$. Then the K -orbit of Δ_0 in $\mathfrak{m}$, $K\Delta_0$, is also a quotient of $K/C_K t_{\mathfrak{m}} \times \Delta_0$. The relations are $(k_1 X) \sim (k_2 X)$ if $X \in \Delta_I$ and $k_1 = k_2 \text{ mod } K_I$. But $K_I = K \cap \mathcal{O}_I$, so these relations are identical to the ones that define the building. $\square$

§ 3. LOOP GROUPS

Let $LG, LG_{\mathbf{C}}$ denote the free loop spaces. Let $G_{\mathbf{C}}$ denote the group of loops which are restrictions of regular maps $\mathbf{C}^* \rightarrow G_{\mathbf{C}}$, and let $L_{\text{alg}}G = L_{\text{alg}}G_{\mathbf{C}} \cap LG$. Thus if we fix an embedding $G_{\mathbf{C}} \subset GL(n, \mathbf{C})$, $L_{\text{alg}}G$ consists of the loops f in LG admitting a finite Laurent expansion $f(z) = \sum_{k=-m}^m A_k z^k$, whereas $L_{\text{alg}}G_{\mathbf{C}}$ consists of the loops f in $LG_{\mathbf{C}}$ such that both f and f^{-1} admit finite Laurent expansions. We will also write $\tilde{G}_{\mathbf{C}}$ for $L_{\text{alg}}G_{\mathbf{C}}$. In fact $\tilde{G}_{\mathbf{C}}$ is the group of points over $\mathbf{C}[z, z^{-1}]$ of the algebraic group $G_{\mathbf{C}}$. Its Lie algebra is the loop algebra $\tilde{g}_{\mathbf{C}}$ of regular maps $\mathbf{C}^* \rightarrow g_{\mathbf{C}}$. The integer m in the above Laurent expansion defines a filtration of $\tilde{G}_{\mathbf{C}}$ by finite dimensional subspaces; we give $\tilde{G}_{\mathbf{C}}$ the corresponding weak topology.

Let P denote the subgroup of $\tilde{G}_{\mathbf{C}}$ consisting of regular maps $\mathbf{C} \rightarrow G_{\mathbf{C}}$ (i.e. maps with nonnegative Laurent expansion, or $G_{\mathbf{C}[z]}$), and let $\tilde{B}$ denote the Iwahori subgroup: $\{f \in P : f(0) \in B^{-}\}$. Finally, let $\tilde{N} = L_{\text{alg}}N_{\mathbf{C}}$, and recall that $\tilde{W}$ can be regarded as a "subgroup" of $\tilde{G}_{\mathbf{C}}$, since $R \leq \text{Hom}(S^1, T) \leq L_{\text{alg}}T$. More precisely, we have $\tilde{N}/T_{\mathbf{C}} = \hat{W}$, and $\tilde{W} \subset \hat{W}$.