§1. Minimal embeddings: définitions and preliminary remarks

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a topologically trivial $\mathbf{P}^1$-bundle over a one-dimensional complex torus, a Hopf surface with abelian fundamental group, or a two-dimensional complex torus. (See also [H-O].) There have been other studies of almost homogeneous surfaces. For example in [Pop] the author describes those which are affine and such that the complement to the open orbit is a finite set of points. In these studies one is primarily interested in the surfaces. In this article, however, we are given the surface and the group, and we are interested in the action.

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§ 1. Minimal embeddings: definitions and preliminary remarks

Let G be a connected algebraic group and let H be an algebraic subgroup.

Definition. An *embedding* of the homogeneous space G/H is a reduced irreducible algebraic variety X endowed with a regular action of G having an open orbit isomorphic to G/H. Two embeddings are equivalent if they are G-isomorphic.

In this paper we study all smooth complete embeddings of B/Γ, where Γ is a finite subgroup of B (any such Γ is cyclic, and two finite subgroups of the same order are conjugate). Since B/Γ is rational and two-dimensional, the underlying variety of such an embedding is a smooth complete rational surface.

Given a smooth complete B/Γ-embedding X with fixed point x, the action of B on X induces an action on $\tilde{X}$, the variety obtained by blowing up x in X, giving $\tilde{X}$ the structure of a B/Γ-embedding. (This is a consequence of the universal property of blowing up. See e.g. [Har], p. 164. See also [O-W], pp. 48-49.) We say that X is a *minimal B/Γ-embedding* if it is not the blow up of another smooth B/Γ-embedding. If X is a minimal model as a variety (that is, if the underlying variety of X is not the blow up of another smooth variety), then clearly X is a minimal embedding. We will now prove the converse.

Lemma 1.1. *Suppose X is a smooth complete surface on which a connected linear algebraic group H acts regularly. Suppose also that X contains an irreducible curve C with a strictly negative self-intersection number. Then C is stable by H.*

Proof. Let $s \in H$. Then since H is connected and the action is regular, sC is linearly equivalent to C. (See e.g. [Gro], p. 5-06, Lemme 1 or [Kam]. See also [O-W], p. 49 and [Ful] for related results.) Thus the intersection number $sC \cdot C$ equals the self-intersection number $C \cdot C$. Since sC is irreducible, the assumption $sC \neq C$ implies that $sC \cdot C$ is non-negative, since these curves intersect in a finite number of points, each counted with positive multiplicity. □

PROPOSITION 1.2. *Suppose X is a minimal B/Γ-embedding. Then it is a minimal model as a variety; that is, X is a rational minimal model.*

Proof. If X is not a minimal model as a variety, then it contains an irreducible curve C isomorphic to $\mathbf{P}^1$ with self-intersection -1. (Castelnuovo criterion. See e.g. [Har] p. 414.) If we apply Lemma 1.1 to the case $H = B$, we see that C is stable by B. By Zariski's Main Theorem (projective-smooth case) (see [Mum], p. 52), the action of B on X induces an action on the surface obtained by blowing down C. Thus this new surface is also a B/Γ-embedding, and X is not a minimal embedding. Also X must be rational, because B/Γ is rational. □

We recall the description of the set of minimal models of rational surfaces (see for example [Har] Section V.2, [Beau] Ch. IV, or [Saf] Ch. V). For any integer $n \geqslant 0$, define $\mathbf{F}_n = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$. (For $k = \mathbf{C}$ these surfaces are known as the Hirzebruch surfaces.) Then $\mathbf{F}_n$ is a ruled surface over $\mathbf{P}^1$. For example, $\mathbf{F}_0 \cong \mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{F}_1$ is the blow up of $\mathbf{P}^2$ in one point. The set of minimal rational models is given by $\mathbf{P}^2$ and $\mathbf{F}_n, n \neq 1$.

Let us review some elementary properties of the surfaces $\mathbf{F}_n$. These facts can be found in the references above. As mentioned above, $\mathbf{F}_n$ is a ruled surface over $\mathbf{P}^1$; that is, it is a $\mathbf{P}^1$-fibre bundle over $\mathbf{P}^1$. We restrict to the case $n \geqslant 1$. Then there is exactly one ruling of $\mathbf{F}_n$, i.e. there is exactly one morphism $\pi_n \colon \mathbf{F}_n \to \mathbf{P}^1$ with fibres isomorphic to $\mathbf{P}^1$. The bundle $\pi_n \colon \mathbf{F}_n \to \mathbf{P}^1$ has a unique section E_n with self-intersection $-n$, and E_n is the only irreducible curve of $\mathbf{F}_n$ with strictly negative self-intersection. The fibres of π_n are all linearly equivalent, and they are the only irreducible curves with self-intersection 0. So any automorphism of $\mathbf{F}_n$ stabilizes E_n and permutes the fibres. Now $\mathbf{F}_n - E_n$ is the total space of the vector bundle $\mathcal{O}(n)$ over $\mathbf{P}^1$. All the sections of $\mathcal{O}(n)$ are linearly equivalent (as divisors of $\mathbf{F}_n$) with self-intersection n.

If one contracts the section E_n of $\mathbf{F}_n, n \geqslant 1$, one obtains a surface X_n (nonsingular if and only if $n=1$) contained in $\mathbf{P}^{n+1}$. In fact X_n is the closure of the affine cone over the n-tuple embedding $\mathbf{P}^1 \to \mathbf{P}^n$ (see [Beau] Ch. IV, Ex. 1 or [G-H], p. 523). That is,

$$X_n = \{(z_0 : s^n : s^{n-1}t : \dots : t^n) \mid z_0, s, t \in k\} \subset \mathbf{P}^{n+1}.$$

The vertex of the cone X_n is $(1:0:\dots:0)$. The image of a general fibre of $\mathbf{F}_n$ in X_n is given by choosing s and t such that $\alpha s = \beta t$ for some $(\alpha : \beta) \in \mathbf{P}^1$.

One can also construct the surfaces $\mathbf{F}_n$ inductively: given $\mathbf{F}_n, n \geqslant 1$, one blows up a point x on E_n and then blows down the strict transform of the fibre containing x to obtain $\mathbf{F}_{n+1}$. The rational map thus obtained from $\mathbf{F}_n$ to $\mathbf{F}_{n+1}$ is sometimes called an *elementary transformation.* (See e.g. [Saf] Ch. V.)

Also, for $n \geqslant 1$, we have an exact sequence

$$1 \to k^* \ltimes H^0(\mathbf{P}^1, \mathcal{O}(n)) \to \operatorname{Aut} \mathbf{F}_n \xrightarrow{\Phi} PGL(2) \to 1$$

where Φ is the restriction of an automorphism to $E_n \cong \mathbf{P}^1$, and k^* acts on $H^0(\mathbf{P}^1, \mathcal{O}(n))$ by multiplication. The kernel of Φ is the subgroup of automorphisms that fix the fibres of π_n. (See [Beau] Ch. V, Ex. 4.)

We define an action of $\operatorname{Aut} \mathbf{F}_n$ on $H^0(\mathbf{P}^1, \mathcal{O}(n))$ as follows. If $\varphi \in \operatorname{Aut} \mathbf{F}_n$ and s is a global section of $\mathcal{O}(n)$, then φs is the section given by $(\varphi s)(x) = \varphi(s(\varphi^{-1}x))$, where $x \in \mathbf{P}^1$ and the action of φ^{-1} on $\mathbf{P}^1$ is given by its action on $E_n \cong \mathbf{P}^1$. Thus $(\varphi s)(\mathbf{P}^1) = \varphi(s(\mathbf{P}^1))$.

LEMMA 1.3. *Let* $\varphi \in \operatorname{Aut} \mathbf{F}_n, n \geqslant 1$; *then the action of* φ *on the vector space* $H^0(\mathbf{P}^1, \mathcal{O}(n))$ *given above is an affine transformation.*

Proof. One has to check that for $s_1, s_2 \in H^0(\mathbf{P}^1, \mathcal{O}(n))$ and $t \in k^*$ we have that $\varphi(ts_1 + (1-t)s_2) = t(\varphi s_1) + (1-t)(\varphi s_2)$. We use that given $x \in \mathbf{P}^1$ the restriction of φ to the fibre $\varphi^{-1}(\pi_n^{-1}x)$ gives an isomorphism

$$k \cong \varphi^{-1}(\pi_n^{-1}x) \xrightarrow{\sim} \pi_n^{-1}x \cong k;$$

this transformation is affine. Now suppose we have s_1, s_2, and t as above; let $s = ts_1 + (1-t)s_2$. Then for any $x \in \mathbf{P}^1$ we have

$$\begin{aligned}(\varphi s)x &= \varphi(s(\varphi^{-1}x)) = \varphi(ts_1(\varphi^{-1}x) + (1-t)s_2(\varphi^{-1}x)) \\ &= t\varphi(s_1(\varphi^{-1}x)) + (1-t)\varphi(s_2(\varphi^{-1}x)) = t(\varphi s_1)x + (1-t)(\varphi s_2)x.\end{aligned}$$

This proves the lemma. □

Thus for $n \geqslant 1$, there is a homomorphism $\operatorname{Aut} \mathbf{F}_n \to \operatorname{Aff}(H^0(\mathbf{P}^1, \mathcal{O}(n))$ given by $\varphi \to (s \to \varphi s)$.

To describe a B/Γ-embedding with underlying variety X, we must give a homomorphism $B \to \operatorname{Aut} X$ such that X has an open orbit B-isomorphic to B/Γ. Two such homomorphisms give rise to equivalent embeddings if and only if they are conjugate.

In the following section we will use the information given here to study the possible B/Γ-embeddings into $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$, and $\mathbf{F}_n$, $n \geqslant 1$.

§ 2. THE MINIMAL B/Γ-EMBEDDINGS

THEOREM 2.1. *Let* Γ *be a finite subgroup of* B, *and let* X *be the projective plane* $\mathbf{P}^2$ *or a rational ruled surface* $\mathbf{F}_n$ *(with* $n \geqslant 0$, *where* $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$*).*

(i) *The number* $\operatorname{emb}(X)$ *of equivalence classes of* B/Γ*-embeddings into* X with at least two fixed points *is*

$$\operatorname{emb}(\mathbf{P}^2) = 2\,, \quad \operatorname{emb}(\mathbf{P}^1 \times \mathbf{P}^1) = 1\,, \quad \textit{and} \quad \operatorname{emb}(\mathbf{F}_n) = n+3,\ n \geqslant 1\,.$$

We call these the "ordinary" embeddings.

(ii) *Moreover, for any such surface* X, *there is exactly one subgroup* Γ *and an "exceptional"* B/Γ*-embedding into* X with only one fixed point *(up to equivalence), and the corresponding order* $\operatorname{ord}(X)$ *of this group* Γ *is*

$$\operatorname{ord}(\mathbf{P}^2) = 4\,, \quad \operatorname{ord}(\mathbf{P}^1 \times \mathbf{P}^1) = 2\,, \quad \textit{and} \quad \operatorname{ord}(\mathbf{F}_n) = 2(n+1),\ n \geqslant 1\,.$$

(iii) *The complement to the open orbit consists of two (for* $\mathbf{P}^2$*) resp. three (for the* $\mathbf{F}_n$*) smooth rational curves, intersecting transversely, except in the "exceptional" case with* $X = \mathbf{P}^2$, *in which case the two curves are tangent.*

(In this theorem we include the case $\mathbf{F}_1$ even though it is not minimal.)

To be more precise, we indicate the form of the complement Z to the open orbit in each case. Also to distinguish the embeddings where Z has the same form, we indicate how the action of B differs on Z. Let U be the unipotent radical of B and T be a maximal torus. (That is, U is the subgroup of elements of B where both eigenvalues are 1, and T can be chosen to be the subgroup of diagonal elements.) Then B is $T \ltimes U$, and the characters of B are the characters of T. We denote the character group of B by $\{\alpha^n : n \in \mathbf{Z}\}$.

Denote by c the order of the group Γ.