4. Properness

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LEMMA 2.1. *Let* A, B *be metric spaces, with* $A \neq \emptyset$ *and* B *connected. Let* $P: A \to B$ *be a continuous map. Assume:*

(i) P *is open,*

(ii) P *is proper, that is, for any compact subset* K *in* $B, P^{-1}(K)$ *is compact. Then* P *is surjective.*

Proof. We only need to prove that $P(A)$ is closed. Let b_0 be a point in $\overline{P(A)}$. Since B is a metric space, there exists a sequence $(b_i)_{i>0}$ in $P(A)$ converging to b_0. The subset $K = \{b_0, b_1, b_2, ...\}$ is compact, hence so is $PP^{-1}(K)$. The latter contains $b_1, ..., b_i, ...$, hence b_0, and it is obviously contained in $P(A)$. Q.E.D.

In order to make use of this lemma, we shall need some inverse function theorem for (i), and some *a priori* estimates for (ii).

3. LOCAL INVERSION

THEOREM 3.1. *Let* X *be a smooth compact manifold,* V *and* W *smooth vector bundles on* X, U *an open set in* $C^{\infty}(X, V)$, *and* $P: U \to C^{\infty}(X, W)$, *a smooth nonlinear elliptic partial differential operator. Let* A *and* B *be LCFC submanifolds of* U *and of* $C^{\infty}(X, W)$ *respectively, such that the restriction* P_A *of* P *to* A, *sends* A *into* B. *Then the Jacobian criterion holds for* P_A, *namely, if the derivative of* $P_A: A \to B$ *is invertible at* $\varphi_0 \in A$, *then* P_A *is a local diffeomorphism near* φ_0.

This is a convenient variant of the Nash-Moser theorem (e.g. [14]) regarding suitable restrictions of elliptic operators. It is established in a separate paper [11] (see also [22]). It relies only on the *classical* (Banach) inverse function theorem combined with *elliptic regularity.*

Remark 3.2. The Nash-Moser theorem has been studied by many authors, see the bibliography below and further references in [14] [15] [25].

4. PROPERNESS

In view of (2), theorem 3.1 implies that P_λ is open. We want to apply lemma 2.1 in order to prove that P_λ is surjective from A_λ to B_λ. Since $P_\lambda(A_\lambda) \neq \emptyset$ (it contains 0), and since B_λ is connected, this amounts to proving that P_λ is *proper.* Let us explain why *a priori* estimates imply properness.

Concerning subsets in A_λ we have

PROPOSITION 4.1. *A subset* S *in* A_λ *is relatively compact in* A_λ *iff its closure* $\bar{S}$ *in* $C^\infty(X)$ *lies inside* A_λ *and* S *is bounded in* $C^\infty(X)$.

This readily follows from Ascoli theorem which implies the well-known fact [12] (p. 231) that in $C^\infty(X)$ (and in any *closed* LCFC submanifold of $C^\infty(X)$, such as B_λ, as well) bounded subsets are relatively compact and vice-versa; hence, *compact* subset of A_λ are nothing but *bounded closed strictly interior* subsets of A_λ. Explicitely, let us state the

COROLLARY 4.2. *A closed subset* S *in* A_λ *is compact if and only if there exists a sequence* (C_i), $i \in \mathbf{N}$, *of positive numbers, such that for any* φ *in* S *the following estimates hold:*

$$\| (g')^{-1} \| =: \sup_X | (g')^{-1} | \leqslant C_0 ,$$

$$\forall i \in \mathbf{N} , \quad \| D^i \varphi \| =: \sup_X | D^i \varphi | \leqslant C_i ,$$

where $|\cdot|$ *denotes some natural norms of tensors in the original metric* g, *and* $D =: (\nabla, \bar{\nabla})$ *is the total covariant differentiation with respect to the metric* g.

Proof. Indeed S is closed and bounded. Moreover, since for $\varphi \in S$,

$$\| (g')^{-1} \| \leqslant C_0$$

all the eigenvalues of $(g')^{-1}$ (which are *positive*) are uniformly bounded *from above*, hence those of g' are uniformly bounded *from below*, in other words:

$$\exists \varepsilon > 0 , \quad \forall \varphi \in S , \quad g' \geqslant \varepsilon g ,$$

or equivalently $\bar{S}$ lies *strictly inside* A_λ. Q.E.D.

In the next sections we will show that if f belongs to some *compact* (i.e. bounded and closed) subset K of B_λ, defined by a sequence (K_i), $i \in \mathbf{N}$, such that $\| D^i f \| \leqslant K_i$, then for $\varphi \in A_\lambda$ satisfying $P_\lambda(\varphi) = f$, the following *a priori* estimates hold:

$$\| \varphi \| \leqslant C_0 , \quad \forall i \in \mathbf{N} , \quad \| D^i \nabla \bar{\nabla} \varphi \| \leqslant C_{i+2} .$$

These estimates imply that P_λ is *proper*, i.e. that $S = P_\lambda^{-1}(K)$ is compact, according to the following

PROPOSITION 4.3. *Let* S *be a closed subset in* A_λ. *Suppose that there exists a sequence* (C_i), $i \in \mathbf{N}$, *such that for any* φ *in* S, *the following estimates hold:*

$$\| \varphi \| \leqslant C_0 , \quad \| P_\lambda(\varphi) \| \leqslant C_0 , \quad \forall i \in \mathbf{N} , \quad \| D^i \nabla \bar{\nabla} \varphi \| \leqslant C_{i+2} .$$

Then S *is compact.*

Proof. The first two estimates imply a uniform estimate

$$| \operatorname{Log} \det (g' g^{-1}) | \leqslant E .$$

The estimate on $\| \nabla \bar{\nabla} \varphi \|$ yields another one:

$$\| g' \| \leqslant F .$$

These two estimates yield

$$\| (g')^{-1} \| \leqslant G .$$

Now from $\| D^i \nabla \bar{\nabla} \varphi \| \leqslant C_{i+2}$ we infer

$$\| D^i \Delta \varphi \| \leqslant \tilde{C}_{i+2}$$

since D and g^{-1} commute (Δ denotes the Laplacian in the metric g). As Δ performs a continuous linear automorphism of the Fréchet space of smooth functions *with zero average* (by Fredholm theory), the Closed Graph Theorem implies the missing estimates. Q.E.D.

Remark 4.4. Actually we have been considering two *gradings* of $C^\infty(X)$ [14]. The usual one, namely the one defined, $\forall u \in C^\infty(X)$, by

$$\| u \|_0 = \sup_X | u | ,$$
$$\| u \|_i = \| u_i \|_{i-1} + \| D^i u \| , \quad i \geqslant 1 ,$$

and another one, well-adapted here since the true unknown is a Kähler metric, defined by

$$\| u \|_0^* = \| u \|_0 , \quad \| u \|_1^* = \| u \|_1 ,$$
$$\| u \|_i^* = \| u \|_{i-1}^* + \| D^{i-2}(\nabla \bar{\nabla} u) \| , \quad i \geqslant 2 .$$

Although it is unnecessary for the purpose of proposition 4.3, it can be shown globally (without Schauder theory) that these two gradings are *tamely* equivalent [14] of degree 2 and base 0 [10] (section 5). Hence, they define the same topology.