# K NON-DYADIC

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## $K$ NON-DYADIC

We begin with a well known lemma, valid for an arbitrary field of characteristic 0.

LEMMA. *Let* $K_0$ *be a subfield of* $K$ *which contains* $\mathbf{Q}^c \cap K$. *Then*

$$S(K) = K \otimes S(K_0)$$

*where* $K \otimes S(K_0)$ *denotes the subgroup of* $B(K)$ *obtained from* $S(K_0)$ *by extension of scalars.*

*Proof.* See Prop. 4.6 in [Y]; a proof of a more general result can be found in [M]. Since the proof is short, we give it here. We can assume $K_0 = \mathbf{Q}^c \cap K$. Let $\beta \in S(K)$ and let $A$ be a Schur algebra in $\beta$, i.e. a simple component of some group algebra $KG$ with Brauer class $[A] = \beta$. Then $A$ is also a direct summand of $K \otimes A_0$ for some simple component $A_0$ of $K_0G$. But the center of $A_0$ is a sub-cyclotomic extension of $K_0$ (see exercise 9.15, [I], e.g.), so is $= K_0$ since it is also contained in $K$. Thus $[A_0] \in S(K_0)$ and $A = K \otimes A_0$. It follows that $S(K) \subseteq K \otimes S(K_0)$ and the reverse inclusion is obvious. □

This lemma allows us to assume, from now on, that $K$ is a sub-cyclotomic extension of $\mathbf{Q}_p$, i.e. a (finite) abelian extension of $\mathbf{Q}_p$.

We shall denote the group of roots of unity of a field $L$ by $\mu(L)$. The subgroup of roots of unity of order a power of $p$, resp. of order relatively prime to $p$, is denoted by $\mu(L)_p$ resp. $\mu(L)_{p'}$. The group of *all* roots of unity, i.e. $\mu(\mathbf{Q}_p^c)$, will be denoted by $\mu$, with $\mu_2$ and $\mu_{2'}$ having the obvious meanings.

*Assume* now that $p$ is odd. Since $\mu(\mathbf{Q}_p)$ is $\cong \mathbf{Z}/p-1$, the root of unity theorem of Benard and Schacher (Th. 6.1, [Y]) and the fact that $B(\mathbf{Q}_p) \cong \mathbf{Q}/\mathbf{Z}$ (see [S], e.g.) imply that $S(\mathbf{Q}_p) \hookrightarrow \mathbf{Z}/p-1$. (In fact this map is an isomorphism). By the theory of central simple algebras over a local field, we can identify $B(K)$ with $H^2(\mathscr{G}(\mathbf{Q}_p^c/K), (\mathbf{Q}_p^c)^*)$, (which we also denote by $H^2(\mathbf{Q}_p^c/K)$). By the Brauer-Witt theorem (Cor. 3.11, [Y]), $S(K)$ is thereby identified with the image of the canonical map

$$H^2(\mathscr{G}(\mathbf{Q}_p^c/K), \mu) \to H^2(\mathscr{G}(\mathbf{Q}_p^c/K), (\mathbf{Q}_p^c)^*), \quad (\mu = \mu(\mathbf{Q}_p^c)),$$

which we denote by $H_c^2(\mathbf{Q}_p^c/K)$. The (cohomological) corestriction map $B(K) \to B(\mathbf{Q}_p)$ carries $S(K)$ into $S(\mathbf{Q}_p)$ since, on the cocycle level, it takes a cocycle $f$ to a cocycle whose values are products of the values of $f$ (see [W], e.g.). Furthermore the corestriction is injective in this case

(p. 175, [S]) and so $S(K)$ is finite — in fact it is a subgroup of $\mathbf{Z}/p-1$. We may therefore choose a primitive $m^{\text{th}}$ root of unity $\varepsilon_m$ so that

$$S(K) = H_c^2(\mathbf{Q}_p(\varepsilon_m)/K).$$

We may also assume that $p \mid m$, i.e. that $\varepsilon_p \in \mathbf{Q}_p(\varepsilon_m)$.

We now show that $\mathbf{Q}_p(\varepsilon_m)$ can be replaced by a field $L$ so that $L/K$ is cyclic and totally tamely ramified. First of all, by Lemma 4.1, [Y], we can assume that $\mathbf{Q}_p(\varepsilon_m)$ is the (disjoint) compositum $UV$ of an unramified extension $U/K$ and a totally ramified extension $V/K$. Since the order of $S(K)$ is relatively prime to $p$, $S(K)$ is the image $H_c^2(\mathbf{Q}_p(\varepsilon_m)/K)'$ of the canonical map

$$H^2(\mathcal{G}(\mathbf{Q}_p(\varepsilon_m)/K), \mu(\mathbf{Q}_p(\varepsilon_m))_{p'}) \to H^2(\mathbf{Q}_p(\varepsilon_m)/K).$$

Since $UV/V$ is unramified, $N_{UV/V}(\mu(UV)_{p'}) = \mu(V)_{p'}$, and it follows from the inflation-restriction sequence (see Lemme 1, [F]) that the inflation

$$H^2(\mathcal{G}(V/K), \mu(V)_{p'}) \to H^2(\mathcal{G}(UV/K), \mu(UV)_{p'})$$

is an isomorphism (since $UV/V$ cyclic implies that $H^2(\mathcal{G}(UV/V), \mu(UV)_{p'}) \cong H^0(\mathcal{G}(UV/V), \mu(UV)_{p'}) = 1$). Thus $S(K) = H_c^2(V/K)'$. Let $L/K$ be the tamely ramified part of $V/K$. Since the $p'$ roots of unity in a local field are the same as the non-zero elements in the residue class field, $\mu(V)_{p'} = \mu(L)_{p'} = \mu(K)_{p'}$ and so $N_{V/L}(\mu(V)_{p'}) = \mu(L)_{p'}$ because $(V:L)$ is a power of $p$. Once again the inflation-restriction sequence shows that $S(K) = H_c^2(L/K)'$.

Consider now the cup product pairing

$$(1) \qquad \smile : \mathcal{G}(L/K) \times H^2(L/K) \to K^*/N_{L/K}L^*.$$

See for example pp. 139-140, [C-F]. It is known that there is a "canonical class" $u_{L/K}$ in $H^2(L/K)$ with the property that the map $\sigma \mapsto \sigma \smile u_{L/K}$ is an isomorphism $\mathcal{G}(L/K) \to K^*/N_{L/K}L^*$. It follows that if $\sigma$ is a generator of $\mathcal{G}(L/K)$, the map

$$\sigma \smile : H^2(L/K) \to K^*/N_{L/K}L^*$$

is also an isomorphism. We wish to identify the image of $H_c^2(L/K)'$ under this map. The cohomology class $[f]$ of the cocycle $f$ has image $\prod_\tau f(\tau, \sigma) \bmod N_{L/K}L^*$ (Lemme 4, p. 186, [S]). Since $\mathcal{G}(L/K)$ is cyclic with generator $\sigma$, every cohomology class with coefficients in an arbitrary $\mathcal{G}(L/K)$-module $A$ is represented by a cocycle $f$ of the form

$$f(\sigma^i, \sigma^j) = \begin{cases} 1 & \text{if} \quad i+j < d\,, \\ a & \text{if} \quad i+j \geqslant d\,. \end{cases}$$

Here $d = |\mathscr{G}(L/K)|$, $0 \leqslant i, j < d$, and $a$ is an arbitrary element of $A^{\mathscr{G}(L/K)}$. If $A = L^*$, it follows that a class in $H^2(L/K)$ is in $H^2_c(L/K)'$ iff it contains such an $f$ with $a \in \mu(K)_{p'}$. Since it is clear that $\sigma \smile [f] = a \bmod N_{L/K}L^*$, we see that the image of $H^2_c(L/K)'$ is

$$\mu(K)_{p'}N_{L/K}L^*/N_{L/K}L^* \cong \mu(K)_{p'}/\mu(K)_{p'} \cap N_{L/K}L^*\,.$$

But it is easy to see that $\mu(K)_{p'} \cap N_{L/K}L^* = N_{L/K}\mu(L)_{p'}$ so we have an isomorphism

$$S(K) = \mu(K)_{p'}/N_{L/K}\mu(L)_{p'}$$

depending only on the choice of $\sigma$.

We now show that the norm residue symbol

$$\nu_K = (\quad, \mathbf{Q}_p^c/K) : K^* \to \mathscr{G}(\mathbf{Q}_p^c/K)$$

induces an isomorphism of $\mu(K)_{p'}/N_{L/K}\mu(L)_{p'}$ onto tor $\mathscr{G}(\mathbf{Q}_p^c/K)$. It is clear that the image of $\mu(K)_{p'}$ is contained in tor $\mathscr{G}(\mathbf{Q}_p^c/K)$. Let $\nu = (\quad, \mathbf{Q}_p^c/\mathbf{Q}_p)$. The diagram

$$\begin{array}{ccc} K^* & \xrightarrow{\nu_K} & \mathscr{G}(\mathbf{Q}_p^c/K) \\ N_{K/\mathbf{Q}_p} \downarrow & & \downarrow \text{incl.} \\ \mathbf{Q}_p^* & \xrightarrow{\nu} & \mathscr{G}(\mathbf{Q}_p^c/\mathbf{Q}_p) \end{array}$$

is commutative (Prop. 10, ch. XIII, [S]). Recall now that $\varepsilon_p \in \mathbf{Q}_p(\varepsilon_m)$. It follows that the tame ramification index of $L/\mathbf{Q}_p$ is $p-1$. Therefore if $L'/\mathbf{Q}_p$ is the maximal unramified subextension of $L/\mathbf{Q}_p$, then $(L:L')$ is a $p$-power multiple of $p-1$. Since $\mu(L)_{p'} = \mu(L')_{p'}$ and $N_{L'/\mathbf{Q}_p}\mu(L')_{p'} = \mu(\mathbf{Q}_p)_{p'} \cong \mathbf{Z}/p-1$, the kernel $\kappa$ of the restriction of $N_{K/\mathbf{Q}_p}$ to $\mu(K)_{p'}$ is $\subseteq N_{L/K}\mu(L)_{p'}$. On the other hand if one factors $N_{K/\mathbf{Q}_p}$ through the tame and unramified closures of $\mathbf{Q}_p$ in $K$, one sees that $N_{L/K}$ on $\mu(K)_{p'}$ is $\varepsilon \mapsto \varepsilon^{e(p^f-1)/(p-1)}$ where $e$ and $f$ are resp. the ramification and inertial indices of $K/\mathbf{Q}_p$. It follows that $\kappa = N_{L/K}\mu(L)_{p'}$ which is equal to $\ker \nu_K|_{\mu(K)_{p'}}$ since $\nu$ is injective.

Now $\nu$ maps the torsion subgroup $\mu(\mathbf{Q}_p) \cong \mathbf{Z}/p-1$ of $\mathbf{Q}_p^*$ onto the torsion subgroup of $\mathscr{G}(\mathbf{Q}_p^c/\mathbf{Q}_p)$. Furthermore an element $a \in \mathbf{Q}_p^*$ is mapped into $\mathscr{G}(\mathbf{Q}_p^c/K)$ iff $a \in N_{K/\mathbf{Q}_p}K^*$. It follows at once that $\nu$ maps $N_{K/\mathbf{Q}_p}\mu(K)_{p'}$ isomorphically onto tor $\mathscr{G}(\mathbf{Q}_p^c/K)$. This proves our main theorem in the case $p$ odd.

## $K$ Dyadic

It would be very nice to have a unified proof for the dyadic and non-dyadic cases along the lines of the one above for the non-dyadic case. However that would require a "deflation" $H_c^2(V/K) \cong H_c^2(L/K)$ to some *cyclic* extension $L/K$ in order that the cup product pairing (1) be non-degenerate on both sides. U. Jannsen has shown that this is impossible in general. Since $H^2(L_1/K) = H^2(L_2/K)$ (when inflated to a common extension) if $(L_1 : K) = (L_2 : K)$, one can try to replace the cyclotomic extension $V/K$ by a some cyclic but possibly non-cyclotomic extension to achieve non-degeneracy. This is done, however, at the expense of losing the identification of $S(K)$ as the subgroup of cyclotomic cocycles. This is essentially what is done in the second half of the following proof.

Since $\mu(\mathbf{Q}_2) = \pm 1$, it follows (as in the non-dyadic case) that $S(K)$ is 1 or $\pm 1$. Thus to prove the theorem it suffices to show that

$$S(K) \neq 1 \Leftrightarrow -1 \in \mathcal{G}(\mathbf{Q}_2^c/K)\,. \tag{2}$$

Before beginning we recall a few facts about Galois groups of $\mathbf{Q}_2^c$. Let $\varepsilon_m$ be a primitive $m^{\text{th}}$ root of unity and write $m = 2^n m'$ where $m'$ is odd. Let $f$ be the smallest integer such that $m' \mid 2^f - 1$. Then if $n \geqslant 2$,

$$\mathcal{G}(\mathbf{Q}_2(\varepsilon_m)/\mathbf{Q}_2) \cong \mathbf{Z}/f \times (\mathbf{Z}/2^n)^* \cong \mathbf{Z}/f \times \mathbf{Z}/2^{n-2} \times \mathbf{Z}/2\,.$$

Taking $\varprojlim$ over $m$ one gets

$$\mathcal{G}(\mathbf{Q}_2^c/\mathbf{Q}_2) \cong \hat{\mathbf{Z}} \times \hat{\mathbf{Z}}_2 \times \mathbf{Z}/2\,. \tag{3}$$

The topological generator 1 of $\hat{\mathbf{Z}}$ is the Frobenius of the maximal unramified extension $\mathbf{Q}_2(\mu_{2'})$ of $\mathbf{Q}_2$. The topological generator 1 of $\hat{\mathbf{Z}}_2$ and the generator 1 of $\mathbf{Z}/2$ are the automorphisms of the field $\mathbf{Q}_2(\mu_2)$ determined by $\varepsilon \mapsto \varepsilon^5$ and $\varepsilon \mapsto \varepsilon^{-1}$ resp. for all $\varepsilon \in \mu_2$ (see e.g. [H], § 4, 5). We shall denote these automorphisms by $\sigma_5$ and $\sigma_{-1}$ resp.

From (3) we get a "primary decomposition"

$$\mathcal{G}(\mathbf{Q}_2^c/\mathbf{Q}_2) \cong \prod_{p \neq 2} \hat{\mathbf{Z}}_p \times (\hat{\mathbf{Z}}_2 \times \hat{\mathbf{Z}}_2 \times \mathbf{Z}/2) \tag{4}$$

since $\hat{\mathbf{Z}} \cong \prod \hat{\mathbf{Z}}_p$. Since $\mathcal{G}(\mathbf{Q}_2^c/K)$ is an open subgroup, one can show that the isomorphism implied in (4) restricts to an isomorphism

$$\mathcal{G}(\mathbf{Q}_2^c/K) \cong \prod_{p \neq 2} k_p \hat{\mathbf{Z}}_p \times C_K \cong \hat{\mathbf{Z}} \times \hat{\mathbf{Z}}_2 \times D_K$$

where $C_K$ is a $\hat{\mathbf{Z}}_2$-submodule of finite index in the component $\hat{\mathbf{Z}}_2 \times \hat{\mathbf{Z}}_2 \times \mathbf{Z}/2$ of (4), $k_p$ is an integer (or a power of $p$) $=$ to 1 for almost all $p$, and $D_K$ is either the trivial group or $<\sigma_{-1}>$.

We now begin the proof of (2). Suppose first that $\sigma_{-1} \notin \mathcal{G}(\mathbf{Q}_2^c/K)$, i.e. that $\mathcal{G}(\mathbf{Q}_2^c/K) \cong \hat{\mathbf{Z}} \times \hat{\mathbf{Z}}_2$. It suffices to show that $H^2(\mathcal{G}(\mathbf{Q}_2^c/K), \mu_2)$ is trivial. Let $K_{nr}$ be the unramified closure of $K$ in $\mathbf{Q}_2^c$. Then $K_{nr} = K(\mu_{2'})$ and $\mathcal{G}(K_{nr}/K) \cong \hat{\mathbf{Z}}$.

Let $C_n$ denote the cyclic group of order $n$.

LEMMA. *Suppose $C_{2^k}$ operates faithfully on $C_{2^h}$. Then $H^n(C_{2^k}, C_{2^h}) = 1$ for all $n \geqslant 1$ except in one case: $k = 1$ and the non-trivial automorphism in $C_2$ inverts the elements of $C_{2^h}$ (i.e. "$C_{2^k} = <\sigma_{-1}>$").*

This is a well-documented fact, although perhaps not exactly in this form (see e.g. [N], 4.8, or the proof of Lemma 2, [L]). By the Herbrand theory for the cohomology of cyclic groups (see e.g. [S], ch. VIII, § 4), it suffices to show that $\hat{H}^0(C_{2^k}, C_{2^h}) = 1$, i.e. that every fixed element is a norm. There is generator of $C_{2^k}$ which acts by raising the elements of $C_{2^h}$ to either the power $5^{2^{h-k-2}}$, or possibly the power $-5$ if $k = h-2$ (again [H], § 4, 5). Then a straightforward calculation leads to the desired result (one uses the fact that $2^{r+2} \parallel (5^{2^r}-1)$ for all $r \geqslant 0$). □

Since

$$\mathbf{Q}_2^c = K_{nr}(\mu_2), H^n(\mathcal{G}(\mathbf{Q}_2^c/K_{nr}), \mu_2) = \varinjlim H^n(\mathcal{G}(L/K_{nr}), \mu(L)_2)$$

where $L$ runs over the fields $K_{nr}(\varepsilon_{2^h})$, and so is trivial by the lemma for $n \geqslant 1$. Thus the inflation-restriction sequence (p. 126, [C-F])

$$1 \to H^2(\mathcal{G}(K_{nr}/K), \mu(K_{nr})_2) \to H^2(\mathcal{G}(\mathbf{Q}_2^c/K), \mu_2) \to H^2(\mathcal{G}(\mathbf{Q}_2^c/K_{nr}), \mu_2) = 1$$

is exact whence the inflation is an isomorphism. But $\mathcal{G}(K_{nr}/K) = \hat{\mathbf{Z}}$ has cohomological dimension 1, so $H^2(\mathcal{G}(K_{nr}/K), \mu(K_{nr})_2)$ is 1, hence $H^2(\mathcal{G}(\mathbf{Q}_2^c/K), \mu_2)$ is also 1 as desired. (I am grateful to U. Jannsen for the foregoing proof).

We now assume that $\sigma_{-1} \in \mathcal{G}(\mathbf{Q}_2^c/K)$. This part of the proof is derived from pp. 540-542 in [J] and pp. 467-468, [L]. (F. Lorenz has asked me to point out that the proof on pp. 465-466 of the latter paper is incomplete — one must show that $\rho$ is the identity on $k$.)

LEMMA 1. *$K(\varepsilon_4)/K$ is ramified of degree 2.*

*Proof.* It is clear that the extension is of degree 2. Suppose it is unramified. Let $q$ be the number of elements in the residue class field of

$K(\varepsilon_4)$. Then $K(\varepsilon_4) = K(\varepsilon_{q-1})$. But $K(\varepsilon_{q-1})$ is left element-wise fixed by $\sigma_{-1}$, which contradicts the fact that $\varepsilon_4$ is *not* left fixed. □

Let $h$ be the smallest integer $\geqslant 2$ such that there is an odd integer $m$ with the property that $L = \mathbf{Q}_2(\varepsilon_{2^h}, \varepsilon_m)$ contains $K$. By replacing $m$ by a suitable multiple, we can suppose that the residue class degree of $L/K$

$$f(L/K) \equiv 0 \ (\mathrm{mod}\ 2^h)\,.$$

Let $\mathcal{G}$ be the Galois group of this extension. We shall construct a Schur class of $K$ using $L/K$. For this we use the following very useful lemma. Let $G$ be a finite abelian group, written as the direct sum of cyclic subgroups:

$$G = C_1 \oplus C_2 \oplus \ldots \oplus C_r$$

where each $C_i$ is of order $c_i$ with generator $\sigma_i$. Let $A$ be a $G$-module, written multiplicatively. Define the operators $\Delta_{\sigma_i}$ and $N_{\sigma_i}$ on $A$ by

$$\Delta_{\sigma_i} a = a^{\sigma_i - 1}, \ N_{\sigma_i} a = a^{1 + \sigma_i + \sigma_i^2 + \ldots + \sigma_i^{c_i - 1}}.$$

LEMMA. *Let* $\gamma$ *be a cohomology class in* $H^2(G, A)$, *and let* $f$ *be a normalized cocycle in* $\gamma$. *Then the elements*

$$\begin{aligned} a_i &= f(\sigma_i, \sigma_i) f(\sigma_i^2, \sigma_i) \ldots f(\sigma_i^{c_i - 1}, \sigma_i)\,, \\ a_{ij} &= f(\sigma_i, \sigma_j)/f(\sigma_j, \sigma_i) \quad (i \neq j) \end{aligned} \tag{5}$$

*satisfy the following relations:*

$$\begin{gathered} \Delta_{\sigma_i} a_j = \begin{cases} 1 & \text{if} \quad i = j \\ N_{\sigma_j} a_{ij} & \text{if} \quad i \neq j \end{cases}, \\ a_{ij} a_{ji} = 1 \ (i \neq j)\,, \quad \Delta_{\sigma_i} a_{jk} \cdot \Delta_{\sigma_j} a_{ki} \cdot \Delta_{\sigma_k} a_{ij} = 1 \ (i, j, k \text{ distinct})\,. \end{gathered} \tag{6}$$

*Conversely if we have elements* $a_i$ *and* $a_{ij}$ *in* $A$ *satisfying (6), then there is a uniquely determined cohomology class* $\gamma$ *in* $H^2(G, A)$ *and a normalized cocycle* $f$ *in* $\gamma$ *bearing the relationship (5) to the* $a_i$ *and* $a_{ij}$.

*Proof.* This is just a restatement of the abelian case of [Z], III, § 8, Theorem 22, in terms of cocycles. See also [Y], pp. 15-19. □

We now apply this to the situation at hand: $G = \mathcal{G}$ and $A = \mu(L)_2 = \langle \varepsilon_{2^h} \rangle$. First of all we note that the restriction $\sigma_1$ of $\sigma_{-1}$ to $L$ is a non-trivial element of $\mathcal{G}$, and that the minimality of $h$ implies that

$K(\varepsilon_4, \varepsilon_m) = K(\varepsilon_{2^h}, \varepsilon_m) = L$ (see e.g. Lemma 3.3, [J]). Since $K(\varepsilon_4)/K$ is ramified and $K(\varepsilon_m)$ is unramified (because $m$ is odd), $\mathcal{G}$ is the direct product of the Galois groups $<\sigma_1>$ of $L/K(\varepsilon_m)$ and $<\sigma_2>$ (say) of $L/K(\varepsilon_4)$ of orders 2 and $f$ respectively. We now choose $a_1 = 1 = a_2$ and $a_{12} = \varepsilon_{2^h} = a_{21}^{-1}$. Then $N_{\sigma_1}a_{21} = \varepsilon_{2^h}^{-1}\varepsilon_{2^h} = 1$ and $N_{\sigma_2}a_{12} = \varepsilon_{2^h}^s$ where, if $\sigma_2(\varepsilon_{2^h}) = \varepsilon_{2^h}^r$,

$$s = 1 + r + r^2 + \dots + r^{f-1}.$$

Since $\sigma_2(\varepsilon_4) = \varepsilon_4$, we have $r \equiv 1 \pmod 4$.

Claim: $s \equiv 0 \pmod{2^h}$. If $\sigma_2(\varepsilon_{2^h}) = \varepsilon_{2^h}$, we choose $r = 1$; then $s = f \equiv 0 \pmod{2^h}$.

Suppose then that $\sigma_2(\varepsilon_{2^h}) \neq \varepsilon_{2^h}$, and write $s = (r^f - 1)/(r-1)$. Now $r = 1 + 2^k a$ where $h > k \geqslant 2$ and $a$ is odd. By induction $r^{2^i} = 1 + 2^{k+i}a_i$ ($a_i$ an odd integer) for all $i \geqslant 0$, whence the claim.

It follows of course that $N_{\sigma_2}a_{12} = 1$. Therefore the above lemma provides a 2-cocycle $f$ with coefficients in $<\varepsilon_{2^h}>$. We now consider it to have coefficients in $L^*$ and so its cohomology class $\gamma = [f]$ is a Schur class in $B(K)$. We shall show that this class is non-trivial, which will finish the proof of the theorem. This will be effected by showing that $\gamma$ is the inflation of a non-trivial Brauer class arising from the extension $K(\varepsilon_m)/K$ — this latter class will not arise from a cyclotomic cocycle but this of course does not matter.

We shall use the crossed-product algebra $A = (L/K, f)$ in order to carry this out. As a vector space over $L$ it has a basis $u_1^i u_2^j$ where $0 \leqslant i < 2$ and $0 \leqslant j < f$, with $u_1^2 = 1 = u_2^f$ and $u_1 u_2 u_1^{-1} u_2^{-1} = \varepsilon_{2^h}$. We replace $u_2$ by $u_2' = \pi u_2$ where $\pi = \varepsilon_4(1-\varepsilon_{2^h})$. The new parameters are

$$a_1' = u_1^2 = 1, \quad a_{12}' = u_1 u_2' u_1^{-1} u_2'^{-1} = 1, \quad a_2' = u_2'^f = N_{\sigma_2}\pi.$$

By (6), $\Delta_{\sigma_1}a_2' = N_{\sigma_2}a_{12}' = 1$ and $\Delta_{\sigma_2}a_2' = 1$, so $N_{\sigma_2}\pi \in K$. Since $u_1$ and $u_2'$ commute with each other, it follows easily that

$$A = (K(\varepsilon_4)/K, 1) \otimes (K(\varepsilon_m)/K, N_{\sigma_2}\pi).$$

The first of these crossed-product algebras is clearly split but the second is *not* split: $\pi$ is a prime element of $L$, so $N_{\sigma_2}\pi$ has order (valuation) $f$ in $K(\varepsilon_4)$ ($L/K(\varepsilon_4)$ is unramified), hence order $^1/_2 f$ in $K$ ($K(\varepsilon_4)/K$ is ramified); but the (non-zero) norms in $K$ from $K(\varepsilon_m)$ are exactly the elements whose order is a multiple of $f$ (since the extension is unramified of degree $f$). Thus $A$ is not split, as desired. □