§2. CONFORMAL COMPACTIFICATIONS AND THEIR TOPOLOGY

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objects with support in the limit set can be obtained. The twistor spaces may provide a natural environment to study theorems about the 3-manifold which rely on properties of the geodesic flow. In particular one could try to prove Mostow's theorem (and Thurston's generalisation of it) along the lines outlined in § 6.

From an analytical study of monopoles it is known that monopoles exist under reasonable conditions. This shows that there are interesting holomorphic bundles on twistor space. Understanding the structure of these will almost certainly reveal a large amount of geometry and analysis associated to the hyperbolic manifold. Finally, properties of the moduli spaces of monopoles which are independent of the metric on the 3-manifold are topological invariants of the 3-manifold. This is related to the work of Donaldson and Casson.

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§ 2. Conformal compactifications and their topology

Let $\bar{M}$ be an oriented, irreducible, atoroidal, compact, three-dimensional manifold with non-empty boundary $\delta\bar{M}$. *Atoroidal* means that every map $T^2 \to \bar{M}$ has a kernel on the level of fundamental groups. For simplicity we shall avoid cusps and thus we assume that:

2.1 either no component of $\delta\bar{M}$ is of genus 1 or $\bar{M} = \bar{D}^2 \times S^1$.

Thurston's uniformization theorem (see Morgan [29]) asserts that there is a complete, geometrically finite, hyperbolic structure on $M = \bar{M} - \delta\bar{M}$. This means that M can be realised as follows (see Bers [7], Maskit [27], Morgan [29], Beardon [6] for background).

Recall that $PSL(2, \mathbf{C}) = SL(2, \mathbf{C})/\{\pm 1\}$ is the isometry group of hyperbolic 3-space H^3, and that the right action of an isometry on $H^3 = SU(2)\backslash SL(2, \mathbf{C})$ extends over the boundary $S^2 \cong \delta H^3$ as an action by a fractional linear transformation of S^2. A *Kleinian group* Γ without cusps is a discrete subgroup of $PSL(2, \mathbf{C})$ all elements of which are loxodromic (i.e. have exactly two fixed points in $\bar{H}^3 = H^3 \cup S^2$), and which acts freely and properly on a non-empty open set $\Omega \subset S^2$ (Felix Klein, the man of the dicrete groups, and Oscar Klein, of the Kaluza-Klein theories mentioned in the introduction, are not the same). Proper means that the map $\Omega \times \Gamma \to \Omega \times \Omega : (x, \gamma) \to (x\gamma, x)$ is proper. Proper actions are well behaved, and a proper free action has a smooth quotient, see Gleason [14].

There is a preferred region $\Omega(\Gamma)$, in which Γ acts properly. Define the *limit set* $\Lambda(\Gamma)$ of the group Γ to be the set of all $y \in S^2$ such that there is a sequence of different elements $\gamma_j \in \Gamma$ and an $x \in S^2$ with $\gamma_j \cdot x \to y$. The *region of discontinuity* $\Omega(\Gamma)$ is the complement $S^2 - \Lambda(\Gamma)$, and Γ acts properly on $\Omega(\Gamma)$. The limit set may be quite wild and has Hausdorff dimension $\dim_H \Lambda(\Gamma) \in [0, 2]$. If no confusion is possible we shall denote $\Omega(\Gamma)$ by Ω and $\Lambda(\Gamma)$ by Λ.

The number of components of Ω is 1, 2 or infinite, and Ω/Γ is a collection of N Riemann surfaces $S_1, \ldots, S_N$, where N is the number of Γ-orbits in the set of components of Ω (N can be infinite). It is well known that the Γ-action on H^3 is proper and that it extends to a proper action on $\bar{H}^3 - \Lambda$; therefore $(\bar{H}^3 - \Lambda)/\Gamma$ is a smooth manifold with boundary $\Omega/\Gamma = \cup_j S_j$.

In order to ensure that $(\bar{H}^3 - \Lambda)/\Gamma$ is compact we introduce another notion. The group Γ is said to be *geometrically finite* iff there is a finitely sided fundamental polyhedron (Maskit [27]) for the Γ-action on H^3. In this case the quotient $M = H^3/\Gamma$ is the interior of a compact, smooth manifold $\bar{M} = (\bar{H}^3 - \Lambda)/\Gamma$ which has boundary $\delta M = \Omega/\Gamma$, now equal to a finite collection of compact Riemann surfaces without boundary. In this case the hyperbolic structure on M is said to be geometrically finite. If $\Gamma = \{e\}$ we have $N = 1$, $S_1 = S^2$, and if Γ is cyclic then $N = 1$, $S_1 = T^2$; in both of these cases Ω is connected. In all other cases every S_j is a surface of genus $\geqslant 2$.

The conjugacy class of Γ in $PSL(2, \mathbf{C})$ is not uniquely determined by M as a smooth manifold; in fact continuous deformations of the complete hyperbolic structure on M can be realized by deforming the embedding $\Gamma \to PSL(2, \mathbf{C})$. Thus the situation is much the same as that for Riemann surfaces, which also admit families of hyperbolic structure (or equivalently complex structures).

As a metric space, M endowed with such a hyperbolic structure is highly non-compact, and the boundary surfaces lie at infinity, i.e. they are the celestial surfaces in M. Following the physical idea of a Kaluza-Klein theory we shall study the fibre bundle $M \times S^1$ over M instead of M itself. Another popular notion in physics is that of a *conformal compactification*: $M \times S^1$ has a natural conformal compactification X without boundary (or X_Γ if we want to indicate the dependence on Γ), i.e. there is an injective conformal immersion $M \times S^1 \to X$ onto a dense subset. To get X we spin $\bar{M}$ around $\delta\bar{M}$, see figure 1, i.e. X is $\bar{M} \times S^1$ with the circles over δM identified to a point. This gives a compact 4-manifold X with an S^1-action. The action is free away from the fixed point set, which is isomorphic to the boundary $\delta\bar{M} = \cup_{j=1,N} S_j$. The normal bundles of the S_j are trivial and of S^1-weight 1. For example take $M \cong S \times \mathbf{R}$ with S a surface. Then X is the compactification of $S \times \mathbf{R} \times S^1 \cong S \times \mathbf{C}^*$, that is $X \cong S \times S^2$, where S^1 acts on S^2 by earth rotation and has two fixed surfaces $S \times \{0, \infty\}$ in X.

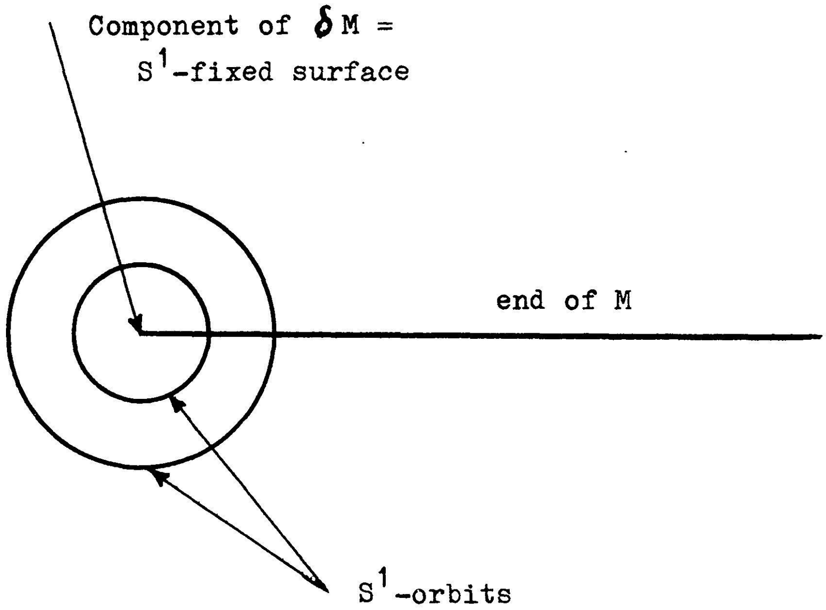

FIGURE 1.

In order to relate the hyperbolic structure on M to a conformal structure on X we proceed more formally. Recall that $H^3 = \{(x, y, t) \in \mathbf{R}^3 ; t > 0\}$ with metric $ds^2 = (dx^2+dy^2+dt^2)/t^2$. It follows that:

2.2 $$i: H^3 \times S^1 \to (\mathbf{R}^2 \oplus \mathbf{R}^2) - (\mathbf{R}^2 \oplus 0) \cong \mathbf{R}^4 - \mathbf{R}^2 \cong S^4 - S^2 :$$
$$((x, y, t), \vartheta) \to (x, y, t\cos\vartheta, t\sin\vartheta)$$

is an orientation preserving, conformal diffeomorphism. The map i intertwines the S^1-action on $H^3 \times S^1$ with rotations in the second summand of $\mathbf{R}^2 \oplus \mathbf{R}^2$. The S^1-action extends to S^4 with fixed point set $S^2 = (\mathbf{R}^2 \oplus 0) \cup \{\infty\} \subset S^4$. This fixed point set corresponds to $\delta H^3 \times S^1$ under:

2.3 $$i' : \bar{H}^3 \times S^1 \to S^4,$$

the continuous extension of i. To get further we shall show that the compactification S^4 of $H^3 \times S^1$ is natural enough to transfer group actions from H^3 to S^4. The maps i and i' are equivariant with respect to the group $S^1 \times PSL(2, \mathbf{C})$, which will act on the right on S^4 by *conformal transformations.* To see this, recall that the $PSL(2, \mathbf{C})$-action on S^4, which is the quaternionic projective line $\mathbf{H}P^1 = \mathbf{H}^*\backslash(\mathbf{H}^2 - \{0\})$ (i.e. divide out the left action of multiplication by invertible quaternions), is by fractional linear transformations:

2.4 $$\left([x, y], \begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) \to [xa + yb, xc + yd]$$

As a result a geometrically finite Kleinian group Γ acts on S^4. The limit set Λ' of the Γ-action on S^4 equals $i'(\Lambda \times S^1)$, so it is contained in the S^1-fixed point set $S^2 \subset S^4$. Clearly Λ' is isomorphic to Λ, and we shall simply identity Λ and Λ'. The restriction:

$$i' : (\bar{H}^3 - \Lambda) \times S^1 \to S^4 - \Lambda$$

is proper, equivariant and surjective. This implies immediately that the Γ-action on $S^4 - \Lambda$ is proper. Since Γ is geometrically finite the quotient $X = (S^4 - \Lambda)/\Gamma$ is compact and without boundary. Finally, the fact that the Γ-action is free ensures that X is smooth and inherits a smooth S^1-action.

The S^1-action is free away from the fixed surfaces S_j, which correspond as conformal surfaces to $\Omega/\Gamma = (\delta H^3 - \Lambda)/\Gamma \cong i'((\delta H^3 - \Lambda) \times S^1)/\Gamma$. It is useful to realise that i and i' induce maps $i : M \times S^1 \to X$ and $i' : \bar{M} \times S^1 \to X$. Summarizing we have proved:

THEOREM 2.1. *Let $\bar{M}$ be an oriented, geometrically finite, complete hyperbolic 3-manifold with non-empty boundary $\delta\bar{M} = \cup S_j$ satisfying* 2.1. *Then $M \times S^1$ has an oriented, smooth conformal compactification X (without boundary) upon which S^1 acts. X is conformally flat and the S^1-action is free away from its fixed surfaces $S_j (j = 1, \ldots, N)$ which correspond as conformal surfaces to the boundary surfaces of $\bar{M}$. The normal bundles N_j of S_j in X are topologically trivial and of S^1-weight* 1. *The hyperbolic structure on M can be reconstructed from X by giving $X - (\cup S_j)$ that metric in the conformal class for which the S^1-orbits have length 2π. Then M is the Riemannian quotient of $X - (\cup_j S_j)$ by S^1.* □

Remark. It is worth pointing out that if one chooses an equatorial embedding of S^n in S^{n+1} then any conformal transformation of S^n extends uniquely to a conformal transformation of S^{n+1} leaving invariant the components $S^{n+1} - S^n$. Thus if Γ is a group acting on S^n then it also acts on S^{n+1}. A Kleinian group can be thought of as a group acting on S^3 with limit set in an equatorial S^2. Theorem 2.1 now says that if Γ is geometrically finite and purely loxodromic then in S^4 we have $\Lambda(\Gamma) \subset S^2 \subset S^4$ and $\Omega(\Gamma)/\Gamma$ is a compact 4-manifold.

The existence of a conformal compactification is not automatic. It is easy to see that $\mathbf{R}^3 \times S^1$ cannot be compactified by adding an S^2 at infinity.

The topology of X is easily described:

PROPOSITION 2.2.

a) $\pi_1(X, m) \cong \pi_1(\bar{M}, m)$ *for* $m \in S_1$ *(S_1 a fixed or boundary surface).*

b) There are natural isomorphisms $H_2(\bar{M}, \delta\bar{M}; \mathbf{Z}) \to H_3(X; \mathbf{Z})$ *and* $H_2(\bar{M}; \mathbf{Z}) \oplus H_1(\bar{M}, \delta\bar{M}; \mathbf{Z}) \to H_2(X; \mathbf{Z})$

The two summands of $H_2(X; \mathbf{Z})$ *(modulo torsion) are isotropic and dual to each other under the intersection form* Q *on* $H_2(X; \mathbf{Z})$*; consequently the signature* $\sigma(X) = 0$, *and* $Q = n$ *times* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, *with* $n = \operatorname{rk} H_2(\bar{M}; \mathbf{Z})$.

c) $\chi(X) = \Sigma_j \chi(S_j)$ *with* χ *denoting the Euler characteristic.*

d) Spin structures on X *exist and the double cover of* S^1 *acts naturally and effectively on any spin structure.*

Proof. a) Of course this is what one expects to be true: $\pi_1(\bar{M} \times S^1, m) \cong \pi_1(\bar{M}, m) \times \mathbf{Z}$, but the $\mathbf{Z}$ factor is killed by shrinking the circles to a point. Formally, remark that a tubular neighbourhood of $\cup S_j$ looks like $(\cup S_j) \times D^2$, and apply the Seifert-van Kampen theorem.

b) Define $j: \bar{M} \to X$ by $j(m) = i'(m, 1)$. Up to S^1-rotation j is defined uniquely by the conformal structure of X. This induces a homomorphism $j_*: H_2(\bar{M}; \mathbf{Z}) \to H_2(X; \mathbf{Z})$. Next remark that if c is a chain in $C_j(\bar{M}, \delta\bar{M}; \mathbf{Z})$ then $i'_*(c \times S^1)$ is a chain in $C_{j+1}(X; \mathbf{Z})$, because the circle shrinking to a point enforces $\delta i'_*(c \times S^1) = 0$. Taking a careful look at the Mayer-Vietoris sequence applied to $(\cup S_j) \times D^2$ and $M \times S^1$ shows that this gives natural isomorphisms as indicated in the proposition. The properties of the intersection form Q follow from the intersection pairing: $H_2(\bar{M}; \mathbf{Z}) \times H_1(\bar{M}, \delta\bar{M}; \mathbf{Z}) \to \mathbf{Z}$.

c) This is easy, using either a and b, or equivariant Lefschetz formulas.

d) Every orientable 3-manifold admits a spin structure, see Stiefel [35]. Give S^1 the spin structure corresponding to the connected double cover, which extends to the disc in $\mathbf{R}^2$; therefore a product spin structure on $M \times S^1$ extends to X. Clearly every spin structure on X arises in this way. The double cover of S^1 is needed to define an action on the spin structure of the orbits in X. □

The spin bundle of H^3 is the Spin(3) $\cong SU(2)$ bundle $SL(2, \mathbf{C}) \to H^3 = SU(2)\backslash SL(2, \mathbf{C})$; thus a spin structure on X is in fact nothing else but a lift of the homomorphism $r: \Gamma \to PSL(2, \mathbf{C})$ which defines M, to a homomorphism $r': \Gamma \to SL(2, \mathbf{C})$.

If N denotes the number of boundary components of $\bar{M}$ (as before) then it follows from the exact sequence of the pair $(M, \delta M)$ that:

2.5 $$\operatorname{rk} H_2(X; \mathbf{Z}) = 2 \cdot \operatorname{rk} H_1(\bar{M}, \delta \bar{M}; \mathbf{Z}) \geqslant 2 \cdot (N-1)$$

Another useful fact to keep in in mind is:

2.6 $$\operatorname{rk} \{\ker (H_1(\delta \bar{M}; \mathbf{Z}) \to H_1(\bar{M}; \mathbf{Z}))\} = \tfrac{1}{2} \cdot \operatorname{rk} H_1(\delta \bar{M}; \mathbf{Z}),$$

which can easily be deduced from Alexander duality and the exact sequence of the pair $(\bar{M}, \delta \bar{M})$.

Examples 2.3. 1) If Γ is the cyclic group generated by $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ with $\lambda \in \mathbf{C}^*$ then the limit set equals $\{0, \infty\}$ in the coordinates on δH^3 supplied by the upper half space model. It is easy to see that $M = H^3/\Gamma = D^2 \times S^1$. To find X, it is easiest to divide out the Γ-action on $S^4 - \Lambda = \mathbf{C}^2 - \{0\}$ which is given by $(z_0, z_1) \to (\lambda^2 z_0, |\lambda|^2 z_1)$. As a result X is a *Hopf surface* diffeomorphic to $S^3 \times S^1$. The S^1-action is given by $(z_0, z_1) \to (z_0, e^{i\vartheta} z_1)$, so the fixed surface is the two-torus $\mathbf{C}^*/<\lambda^{2k}>$.

2) If Γ is *Fuchsian*, i.e. $\Gamma \subset PSL(2, \mathbf{R})$, then H^2/Γ is a compact Riemann surface without boundary S of genus $\geqslant 2$ with metric ds^2. The 3-manifold M is diffeomorphic to $\mathbf{R} \times S$ with metric given by $dl^2 + \cosh^2 l \cdot ds^2$. Clearly it follows that X must be diffeomorphic to $S^2 \times S$. A little computation shows that X is even conformally equivalent to $S^2 \times S$. Thus X is conformally equivalent to the Kähler surface $\mathbf{CP}^1 \times S$.

From the point of view of Kleinian groups, we remark that Ω is the union of two round discs which are both invariant under Γ. The limit set is a smooth circle.

3) A Kleinian group Γ which is not itself Fuchsian, but which contains a Fuchsian subgroup Γ_0 of index two is said to be an *extended Fuchsian*

group. For details see Maskit [28]. The limit sets $\Lambda(\Gamma_0)$ and $\Lambda(\Gamma)$ are equal, and any $\gamma \in \Gamma - \Gamma_0$ swaps the two components of Ω. Such an element γ also gives rise to a fixed-point-free, orientation reversing involution σ of S (compare 2), and one deduces from this that $M \cong H^3/\Gamma$ is a nontrivial $\mathbf{R}$-bundle over S/σ. Remark that $\delta\bar{M} \cong S$.

A standard way to get more interesting 3-manifolds is through the *Klein-Maskit combination theorems* (Maskit [27], Morgan [29]). We shall explain how the simplest of these relates to the 4-manifolds involved. Let Γ_0 and Γ_1 be geometrically finite groups without cusps and $M_j = H^3/\Gamma_j$. Every pair of points $x_j \in \delta\bar{M}_j$ has neighbourhoods K_j in M_j isometric to a hyperbolic half space i.e. to a component of $H^3 - H^2$. The $H_j^2 = \delta K_j - \delta\bar{M}_j$ are embedded in M_j and $\delta H_j^2 \cap \bar{M}_j = H_j^2 \cap \delta\bar{M}_j$ are circles which bound discs in $\delta\bar{M}_j$. Define $M = M_0 \# M_1$ to be $M_0\backslash K_0 \cup_\rho M_1\backslash K_1$, where ρ is an isometry $\delta K_0 \to \delta K_1$. The metric structure of $M = M_0 \# M_1$ depends on ρ, the choice of x_j and the choice of the half spaces K_j. M is called a *boundary connected sum* of M_0 and M_1.

The *first combination theorem* expresses the fact that $M = H^3/\Gamma$ with Γ a Kleinian group which is isomorphic to the free product of Γ_0 and Γ_1. In $PSL(2, \mathbf{C})$ the group Γ is generated by Γ_0 and $g\Gamma_1 g^{-1}$ for a suitable $g \in PSL(2, \mathbf{C})$. It is easy to verify this.

Reverting to the 4-manifolds, we see that we are identifying, by S^1-equivariant conformal maps, balls B_j around the points x_j in the fixed surfaces. Thus X_Γ equals $X_{\Gamma_0} \# X_{\Gamma_1}$ with $\#$ now denoting a conformal connected sum. Summarizing we get:

PROPOSITION 2.4. *If Γ is the Kleinian group corresponding to a boundary connected sum of H^3/Γ_0 and H^3/Γ_1 then Γ is a Kleinian group such that X_Γ is the S^1-equivariant conformal connected sum of X_{Γ_0} and X_{Γ_1} at points in the fixed surfaces.*

Example 2.5. A *classical Schottky group* Γ of genus g is a free product of g cyclic groups (compare example 2.3 (1)), formed as in the gluing construction described before proposition 2.4. The 3-manifold M_Γ is a handlebody of genus g, and by proposition 2.4, X_Γ equals the connected sum $(S^3 \times S^1)^{\# g}$. In fact if Γ is any geometrically finite free Kleinian group without cusps, then H^3/Γ is a handlebody; this follows from standard results in 3-manifold topology (see Hempel [17]). We shall refer to such free groups as Schottky groups.