

# 6. Coordinate free tensor calculus

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## 5. A PRIORI ESTIMATES: THE ORIGINAL WAY

According to proposition 4.3 we must prove now that, given any sequence of positive real numbers  $(K_i), i \in \mathbf{N}$ , there exists a sequence  $(C_i)$  such that

$$\forall i \in \mathbf{N}, \quad \| D^i P_\lambda(\varphi) \| \leq K_i$$

implies

$$\| \varphi \| \leq C_0, \quad \forall i \in \mathbf{N}, \quad \| D^i \nabla \bar{\nabla} \varphi \| \leq C_{i+2}.$$

These are *a priori* estimates of order zero, two, three, and so on ... In case  $\lambda > 0$ , the  $C^0$  estimate is straightforward [2]. In case  $\lambda = 0$ , it becomes very tricky; proofs simpler than Yau's original one [24] (p. 352-359), based on the idea of uniformly estimating the  $L^p(dX_g)$  norms of  $\varphi$ , may be found in [16] (dimension 2), [3] [21] and [4] (p. 148-149).

Estimates of order two and three are carried out by means of tensor calculus and of the Maximum Principle (for elliptic equations) [20] applied to *suitable* test functions. Though it is not everywhere clear in [21] [24], it is worth noting that the computations can be written intrinsically, i.e. without any reference to a *particular* system of coordinates (e.g. [2]), or even *coordinate free* (see section 6 below).

Further regularity is then recovered by Schauder theory e.g. [5] (lemma 1). In the sequel, we show how further estimates can be carried out instead, *just going ahead with coordinate free tensor calculus*. This occurs actually for any fully nonlinear second order elliptic equation on a compact Riemannian manifold, via a straightforward imitation of the device below.

*Remark 5.1.* It follows from the  $C^2$  *a priori* estimates that the metrics  $g'$  are *a priori* uniformly equivalent to the original metric  $g$  (see e.g. [3], p. 75).

## 6. COORDINATE FREE TENSOR CALCULUS

Even coordinate free tensor calculus needs indices. Usually these indices refer to a *local* frame. Another way is to view these indices *globally* as labelling copies of the holomorphic and antiholomorphic tangent and cotangent bundles. From this point of view, a tensor written with indices is a section of the tensor product of a family of bundles indexed by an *unordered* set of indices (disregarding those indices subject to the summation convention).

We extend the summation convention as follows: we will be concerned only with lower indices. If a letter occurs twice, it refers to a contraction, which is taken with respect to  $g$  or to  $g'$  according to whether the letter occurs with a bar or with a prime. So,

$$T_{\dots a \dots \bar{a} \dots} \text{ stands for } g^{a\bar{b}} T_{\dots a \dots \bar{b} \dots}, \text{ while}$$

$$T_{\dots a \dots a' \dots} \text{ stands for } g'^{a\bar{b}} T_{\dots a \dots \bar{b} \dots} .$$

As usual if  $T_{a\dots l}$  is a tensor, further lower indices refer to covariant differentiation (with respect to  $g$ ); so,

$$T_{a\dots lm} \text{ stands for } \nabla_m T_{a\dots l}, \text{ while}$$

$$T_{a\dots l\bar{m}} \text{ stands for } \bar{\nabla}_{\bar{m}} T_{a\dots l} .$$

Our indices will be latin letters; greek letters will denote multi-indices. If  $\alpha$  is a multi-index,  $\bar{\alpha}$  will denote the *conjugate* multi-index (for instance if  $\alpha = \bar{a}\bar{b}\bar{c}$ , then  $\bar{\alpha} = \bar{a}\bar{b}\bar{c}$ ), while  $|\alpha|$  denotes its length. We shall say that  $\alpha$  is *mixed* if its length is at least two and, among the first two letters, *exactly* one has a bar.

The notations  $D, \nabla, \bar{\nabla}, \parallel, \parallel$ , were introduced in section 4.

*Remark 6.1.* Since covariant differentiation (with respect to  $g$ ) and contraction with respect to  $g'$  *do not* commute, we observe that, for instance, the difference (recall  $g' = g + \nabla\bar{\nabla}\phi$ )

$$(3) \quad \phi_{aa'ab} - (\phi_{aa'\alpha})_b \equiv \phi_{ac\alpha} \phi_{a'c'b}$$

does not vanish.

## 7. HIGHER ORDER A PRIORI ESTIMATES: GENERALITIES

We want to prove by induction,

PROPOSITION 7.1. *Given  $n \geq 4$ , a sequence  $(K_i), i \in \mathbf{N}$ , and a finite sequence  $C_0, \dots, C_{n-1}$ , there exists  $C_n$  such that:*

$$\|\phi\| \leq C_0, \quad \forall i = 0, \dots, n-3, \quad \|D^i \nabla \bar{\nabla} \phi\| \leq C_{i+2}$$

and  $\forall i \in \mathbf{N}, \quad \|D^i P_\lambda(\phi)\| \leq K_i,$

*implies*

$$\|D^{n-2} \nabla \bar{\nabla} \phi\| \leq C_n .$$