# 8. A PRIORI ESTIMATES OF ORDER FOUR

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$$
\begin{array}{lll}
T_{3,3} + T_2 & \textit{when} & n = 2\,, \\
T_{4,3} + T_{3,3,3} + T_3 & \textit{when} & n = 3\,, \\
T_5 + T_{4,4} + T_4 & \textit{when} & n = 4\,, \\
T_{n+1} + T_n & \textit{when} & n \geqslant 5\,.
\end{array}
$$

*Proof.* The cases $n = 2, 3, 4, 5$, must be checked bare-handed. There is no difficulty. Then, for $n \geqslant 5$, one can proceed by induction on $n$. Indeed assume,

$$\varphi_{aa'\alpha} = T_{n+1} + T_n \text{ mod. } E_{n-1}\,, \qquad \text{for some } n = |\,\alpha\,| \geqslant 5\,.$$

Recall formula (3) and lemma 7.3; differentiating once the above equality yields

$$\varphi_{aa'\alpha b} = (T_{n+1} + T_n)_b + \varphi_{ac\alpha}\,\varphi_{a'c'b} = T_{n+2} + T_{n+1} \text{ mod. } E_n\,,$$

since $|\,ac\alpha\,| = n + 2$. The same is true with $\bar{b}$ instead of $b$. Q.E.D.

*Remark* 7.7. The preceding lemma offers a perspective which brings some light on the type of difficulties to be expected for carrying out *a priori* estimates of each order. In particular, one may anticipate that a special step should be required for $n = 4$ (in order to kill the effect of the term $T_{4,4}$) and that the same (simpler) procedure should then apply, arguing by iteration, for any $n \geqslant 5$.

Notice also that the hardest case appears to be $n = 3$. Indeed, following Calabi [8] one must guess the very special *coercive* functional [1] [24]

$$S_{3,3} = \varphi_{ab'c}\,\varphi_{a'bc'}\,,$$

perform a careful calculation of $\Delta'(S_{3,3})$ and use either the Maximum Principle [24] or a recurrence on $L^p(dX_{g'})$ norms of $S_{3,3}$ [1]. The *approximate* tensor calculus which we may conveniently use hereafter would not be effective for the case $n = 3$.

## 8. A PRIORI ESTIMATES OF ORDER FOUR

In order to prove 7.1 with $n = 4$, we consider the functional:

$$S_{4,4} = \varphi_{a\bar{b}cd}\,\varphi_{\bar{a}b\bar{c}\bar{d}} + \varphi_{a\bar{b}\bar{c}d}\,\varphi_{\bar{a}bc\bar{d}}\,.$$

It is enough to estimate $S_{4,4}$ since it is *coercive.* Let us compute $-\Delta'(S_{4,4})$. One readily obtains:

$$-\Delta'(S_{4,4}) = T_{6,4} + T_{5,5} \quad (\text{mod. } E_3)\,,$$

where $T_{5,5}$ is *coercive*, while the sixth order derivatives in $T_{6,4}$ occur through $\varphi_{a\bar{b}\alpha cc'}$ with $|\alpha| = 2$.

In view of 7.4 and 7.6, after bringing the indices $cc'$ in first position, we get

$$(4) \qquad -\Delta'(S_{4,4}) = T_{5,5} + T_{5,4} + T_{4,4,4} + T_{4,4} + T_4 \qquad (\text{mod. } E_3)$$

where $T_{5,5}$ is the *coercive* term from above.

As expected in remark 7.7, in order to control the term $T_{4,4,4}$, we need to consider instead of $S_{4,4}$ another functional, namely:

$$\theta = S_{4,4} \exp(\varepsilon\, \varphi_{a\bar{b}c}\, \varphi_{\bar{a}b\bar{c}}),$$

where $\varepsilon$ is a constant to be chosen later on. Then we compute the quantity

$$Q = -(\Delta'\theta)\exp(-\varepsilon\varphi_{a\bar{b}c}\,\varphi_{\bar{a}b\bar{c}});$$

and we easily find

$$Q = -\Delta'(S_{4,4}) + \varepsilon T_{4,4,4,4} + \varepsilon^2 T'_{4,4,4,4} + \varepsilon T_{5,4,4} \qquad (\text{mod. } E_3),$$

where $T'_{4,4,4,4}$ is a square and where

$$T_{4,4,4,4} = S_{4,4}(\varphi_{a\bar{b}cd}\,\varphi_{\bar{a}b\bar{c}d'} + \varphi_{a\bar{b}cd'}\,\varphi_{\bar{a}b\bar{c}d}).$$

So there exists a constant $c_1$ such that (see remark 5.1),

$$(S_{4,4})^2 \leqslant c_1\, T_{4,4,4,4}.$$

Furthermore we may choose constants $c_i$ such that,

$$|T_{5,4,4}| \leqslant c_2\, S_{4,4}(T_{5,5})^{\frac{1}{2}}, \quad |T_{5,4}| \leqslant c_3(T_{5,5}\, S_{4,4})^{\frac{1}{2}},$$

$$|T_{4,4,4}| \leqslant c_4(S_{4,4})^{\frac{3}{2}}, \quad |T_{4,4}| \leqslant c_5 S_{4,4}, \quad |T_4| \leqslant c_6(S_{4,4})^{\frac{1}{2}}.$$

By splitting $T_{5,5}$ in its two halves and by putting each half together with $T_{5,4,4}$ and $T_{5,4}$ respectively, one obtains:

$$Q \geqslant \left(\frac{\varepsilon}{c_1} - \frac{1}{2}\varepsilon^2 c_2^2\right)(S_{4,4})^2 - c_4(S_{4,4})^{\frac{3}{2}} - \left(c_5 + \frac{1}{2}c_3^2\right) S_{4,4} - c_6(S_{4,4})^{\frac{1}{2}}.$$

Now $\varepsilon$ must be chosen small enough in order for the coefficient of $(S_{4,4})^2$ to be *strictly* positive: $\varepsilon \in (0, (2/c_1\, c_2^2))$.

To complete the proof, one argues that $Q(z_0) \leqslant 0$ at a point $z_0 \in X$ where $\theta$ assumes its *maximum* on $X$, which implies

$$S_{4,4}(z_0) \leqslant c_7 ,$$

for some controlled constant $c_7$, and anywhere else on $X$, since $\theta \leqslant \theta(z_0)$ and $\| D\nabla\bar{\nabla}\varphi \| \leqslant C_3$, one infers that:

$$S_{4,4} \leqslant c_7 \exp(2\varepsilon C_3) .$$

## 9. A PRIORI ESTIMATES OF ORDER FIVE AND MORE

Here, in order to prove 7.1 with $n \geqslant 5$, we consider the functional:

$$S_{n,n} = \frac{1}{2} \sum_{|\alpha| = n-2} \varphi_{a\bar{b}\alpha} \varphi_{\bar{a}b\bar{\alpha}}$$

(the coefficient $\frac{1}{2}$ appears for both definitions of $S_{4,4}$ to agree).

Again $S_{n,n}$ is *coercive* and we compute in a similar way,

$$-\Delta'(S_{n,n}) = T_{n+2,n} + T_{n+1,n+1} \quad (\text{mod. } E_{n-1}) ,$$

where $T_{n+1,n+1}$ is *coercive.* As for $T_{n+2,n}$, proceeding as in the previous section, we find:

$$T_{n+2,n} = T_{n+1,n} + T_{n,n} + T_n \quad (\text{mod. } E_{n-1}) .$$

Hence,

$$-\Delta'(S_{n,n}) = T_{n+1,n+1} + T_{n+1,n} + T_{n,n} + T_n \quad (\text{mod. } E_{n-1}) ,$$

with $T_{n+1,n+1}$ coercive. Changing $n$ into $(n-1)$, for $n \geqslant 6$, yields *still modulo* $E_{n-1}$

$$-\Delta'(S_{n-1,n-1}) = T'_{n,n} + T'_n \quad (\text{mod. } E_{n-1}) .$$

In view of formula (4) of the preceding section, this holds for $n = 5$ as well. From the *coercivity* of $T'_{n,n}$ we may choose constants $c_i > 0$, such that

$$-\Delta'(S_{n-1,n-1}) \geqslant c_1 S_{n,n} - c_2(S_{n,n})^{\frac{1}{2}} - c_3 .$$

Moreover we may choose constants $c_i$ such that

$$| T_{n+1,n} | \leqslant 2c_4(T_{n+1,n+1} S_{n,n})^{\frac{1}{2}} , \quad | T_{n,n} | \leqslant c_5 S_{n,n} , \quad | T_n | \leqslant c_6(S_{n,n})^{\frac{1}{2}} ,$$

$$\text{and} \quad c_1 c_7 > c_4^2 + c_5 .$$