

## §2. Torsions of chain complexes and manifolds

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

### **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

The non-trivial case of Theorem 2 is the case  $l_1 = l_2 = \dots = l_{n-1} = 0$ : otherwise  $u(L) \geq u$  so that  $\Delta_{u-1}(L) = 0$  and we may put  $\lambda = 0$ .

The proof of Theorems 1, 2 goes along the same lines as the proof of the formula (2) given in [4]. These proofs are based on a relationship between the Alexander polynomials and Reidemeister-type torsions, established in [4]. This relationship is recalled in § 2. In § 3 several easy algebraic lemmas are proved. Theorems 1, 2 are proved in § 4.

This research was completed while the author was visiting the University of Geneva. I thank the staff of the Mathematical Department of the University and especially professors J.-C. Hausmann and M. Kervaire for their hospitality.

## § 2. TORSIONS OF CHAIN COMPLEXES AND MANIFOLDS

2.1. THE TORSION OF A CHAIN COMPLEX (see [3]). Let  $Q$  be a field. If  $a = (a_1, \dots, a_n)$  and  $b = (b_1, \dots, b_n)$  are two bases of a  $Q$ -module then  $a_i = \sum_{j=1}^n c_{i,j} b_j$  where  $(c_{i,j})$  is a non-singular  $n \times n$ -matrix over  $Q$ ; the determinant  $\det(c_{i,j}) \in Q \setminus 0$  is denoted by  $[a/b]$ .

Let  $C = (C_m \rightarrow \dots \rightarrow C_0)$  be a chain  $Q$ -complex. Suppose that each  $Q$ -module  $C_i$  is finite dimensional with a preferred basis  $c_i$  and each  $Q$ -module  $H_i(C)$  also has a preferred basis  $h_i$ . (The case  $C_i = 0$  or  $H_i(C) = 0$  is not excluded; by definition the zero module has the empty basis.) In this setting one defines the torsion  $\tau(C) \in Q$  as follows. For each  $i = 1, 2, \dots, m$  choose a sequence  $b_i = (b_1^i, \dots, b_{r_i}^i)$  of elements of  $C_i$  such that  $\partial_{i-1}(b_i) = (\partial_{i-1}(b_1^i), \dots, \partial_{i-1}(b_{r_i}^i))$  is a basis in  $\text{Im}(\partial_{i-1}: C_i \rightarrow C_{i-1})$ . For each  $i = 0, 1, \dots, m$  choose a lifting  $\tilde{h}_i$  of the basis  $h_i$  to  $\text{Ker } \partial_{i-1}$ . The combined sequence  $\partial_i(b_{i+1})\tilde{h}_i b_i$  is a basis in  $C_i$ . (It is understood that  $b_0 = \emptyset$  and  $b_{m+1} = \emptyset$ ). Put

$$(3) \quad \tau(C) = \prod_{i=0}^m [\partial_i(b_{i+1})\tilde{h}_i b_i / c_i]^{\varepsilon(i)}$$

where  $\varepsilon(i) = (-1)^{i+1}$ . Clearly,  $\tau(C) \in Q \setminus 0$ . It is easy to verify that  $\tau(C)$  does not depend on the choice of  $b_i$  and  $\tilde{h}_i$ .

(Note that the torsion of  $C$  defined in Milnor's survey article [3] equals  $\pm \tau(C)^{-1} \in Q / \pm 1$  and that Milnor uses the additive notation for the multiplication in  $Q \setminus 0 = K_1(Q)$ .)

2.1.1. LEMMA (multiplicativity of torsion). *Let  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$  be a short exact sequence of  $m$ -dimensional chain complexes over a field  $Q$ .*

Suppose that for all  $i = 0, 1, \dots, m$  the modules  $C_i, C'_i, C''_i$  are provided with preferred bases  $c'_i, c_i, c''_i$  which are compatible, in the sense that  $[c'_i c''_i / c_i] = \pm 1$ . Suppose that for all  $i = 0, 1, \dots, m$  the homology modules  $H_i(C), H_i(C'), H_i(C'')$  are provided with preferred bases. Let  $\mathcal{H}$  be the homology sequence of the sequence  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ :

$$\mathcal{H} = (H_m(C') \rightarrow H_m(C) \rightarrow \dots \rightarrow H_0(C) \rightarrow H_0(C'')).$$

Consider  $\mathcal{H}$  as an acyclic based chain complex over  $Q$ . Then  $\tau(C) = \pm \tau(C')\tau(C'')\tau(\mathcal{H})$ .

For a proof see [3].

2.2. THE TORSION  $\omega$ . Let  $M$  be an orientable compact smooth manifold of odd dimension  $m$  with  $\text{rg } H_1(M) \geq 1$ . Denote the free abelian group  $H_1(M)/\text{Tors } H_1(M)$  by  $G$ . Denote the fraction field of the group ring  $\mathbf{Z}[G]$  by  $Q$ . Provide  $Q$  with the involution  $q \mapsto \bar{q}$  which sends  $g \in G$  to  $g^{-1}$ . The field  $Q$  defines via the natural homomorphism  $\mathbf{Z}[\pi_1(M)] \rightarrow Q$  a system of local coefficients on  $M$ . We shall denote this system by the same symbol  $Q$ . Assume that  $H_*(\partial M; Q) = 0$ . In this setting one can consider a torsion-type invariant  $\omega(M)$  of  $M$  which is "an element of  $Q \setminus 0$  defined up to multiplication by  $\pm gq\bar{q}$  with  $g \in G$  and  $q \in Q \setminus 0$ " (see [4]).

Recall the definition of  $\omega(M)$  given in [4, § 5]. Let  $\tilde{M} \rightarrow M$  be the regular covering of  $M$  corresponding to the kernel of the natural homomorphism  $\pi_1(M) \rightarrow G$ . Fix a  $C^1$ -triangulation of  $M$  and the induced  $G$ -equivariant triangulation of  $\tilde{M}$ . Choose over each simplex of the (fixed) triangulation of  $M$  a simplex of the triangulation of  $\tilde{M}$ . These simplices in  $\tilde{M}$  being arbitrarily oriented and ordered determine "natural" bases of the modules of the simplicial chain  $\mathbf{Z}[G]$ -complex  $C_*(\tilde{M}; \mathbf{Z})$ . These bases induce "natural"  $Q$ -bases in the chain  $Q$ -complex

$$C = Q \otimes_{\mathbf{Z}[G]} C_*(\tilde{M}; \mathbf{Z}).$$

For all  $i = 0, 1, \dots, m$  choose an arbitrary  $Q$ -basis  $h_i$  in  $H_i(M; Q) = H_i(C)$ . Denote by  $\tau(C, h_0, \dots, h_m)$  the torsion of  $C$  with respect to the bases in chain modules constructed above and the bases  $h_0, h_1, \dots, h_m$  in homology. Since  $H_*(\partial M; Q) = 0$  the semi-linear intersection form  $H_i(M; Q) \times H_{m-i}(M; Q) \rightarrow Q$  is non-singular. Let  $v_i$  be the matrix of this form regarding the bases  $h_i$  and  $h_{m-i}$ . Put

$$d = \tau(C, h_0, h_1, \dots, h_m) \prod_{i=0}^r (\det v_i)^{-\varepsilon(i)} \in Q \setminus 0$$

where  $r = (m-1)/2$  and  $\varepsilon(i) = (-1)^{i+1}$ . It is easy to show that under a different choice of natural bases and bases  $h_0, h_1, \dots, h_m$  the element  $d$  is replaced by  $\pm gq\bar{q}d$  with  $g \in G, q \in Q \setminus 0$ . Thus the set  $\{\pm gq\bar{q}d \mid g \in Q \setminus 0\} \subset Q$  does not depend on the choice of bases. It also does not depend on the choice of triangulation in  $M$ . It is this set which is  $\omega(M)$ .

An explicit formula established in [4] enables us to calculate  $\omega(M)$  in terms of the orders of  $\mathbf{Z}[G]$ -modules  $H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; \mathbf{Z}), H_*(\tilde{M}) = H_*(\tilde{M}; \mathbf{Z})$  and related modules. (The notion of the order of a module is recalled in Sec. 3.1.) Denote by  $J$  the image of the inclusion homomorphism  $H_r(\partial\tilde{M}) \rightarrow H_r(\tilde{M})$  where  $r = (m-1)/2$ . Then up to multiples of type  $q\bar{q}$  with  $q \in Q \setminus 0$

$$(4) \quad \omega(M) = \text{ord}(\text{Tors}_{\mathbf{Z}[G]} H_r(M, \partial M)) (\text{ord } J)^{\varepsilon(r)} \prod_{i=0}^{r-1} [\text{ord } H_i(\partial M)]^{\varepsilon(i)}$$

(see [4, Theorem 5.1.1]). Note that the equalities  $Q \otimes_{\mathbf{Z}[G]} H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; Q) = 0$  imply that  $H_*(\partial\tilde{M})$  and  $J$  are torsion  $\mathbf{Z}[G]$ -modules. Therefore  $\text{ord } H_i(\partial\tilde{M})$  and  $\text{ord } J$  are non-zero elements of  $\mathbf{Z}[G]$ .

We shall apply formula (4) in the case where  $M$  is the exterior of an  $n$ -component link  $K \subset S^m$  with odd  $m$ . The condition  $H_*(\partial M; Q) = 0$  is always fulfilled in this case. Here the field  $Q$  is canonically identified with the field of rational functions of  $n$  variables  $Q_n = Q(t_1, \dots, t_n)$ . Thus  $\omega(M) \subset Q_n$ . If  $m \geq 5$  then (4) implies that

$$\Delta(K)(t_1, \dots, t_n) \cdot \prod_{i=1}^n (t_i - 1) \subset \omega(M).$$

If  $m = 3$  then there exists a unique subset  $\alpha = \alpha(K)$  of the set  $\{1, 2, \dots, n\}$  such that

$$\Delta_{u(K)}(K)(t_1, \dots, t_n) \cdot \prod_{i \in \alpha} (t_i - 1) \subset \omega(M).$$

For proofs and details consult [4, § 5].

### § 3. ALGEBRAIC LEMMAS

3.1. PRELIMINARY DEFINITIONS. For a finitely generated module  $H$  over a (commutative) domain  $R$  we denote by  $\text{rk}_R H$  or, briefly, by  $\text{rk } H$  the integer  $\dim_Q(Q \otimes_R H)$  where  $Q = Q(R)$  denotes the field of fractions of  $R$ . For a  $R$ -linear homomorphism  $f: H \rightarrow H'$  we put  $\text{rk } f = \text{rk}_R f(H)$ . Note that if  $\bar{R}$  is the localization of  $R$  at some multiplicative system then  $Q(\bar{R}) = Q(R)$  and therefore the (exact) functor  $(H \mapsto \bar{R} \otimes_R H, f \mapsto \text{id}_{\bar{R}} \otimes f)$