

Remarks

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **16.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

REMARKS

There are several proofs and several formulations of this result ($S(K)$ when K is dyadic) in the literature. We shall briefly indicate why these formulations are, with one exception, equivalent to the above one.

1. T. Yamada, [Y], p. 88. One formulation of Yamada's theorem is that $S(K)$ is non-trivial iff there is a root of unity ζ such that the inertia group of the extension $\mathbf{Q}_2(\zeta)/K$ is non-cyclic. The inertia group of $\mathbf{Q}_2(\zeta)/K$ is the image of the inertia group of $\mathcal{G}(\mathbf{Q}_2^c/K)$, namely $\mathcal{G}(\mathbf{Q}_2^c/K_{nr})$. The latter group is of the form $\hat{\mathbf{Z}}_2 \times (\mathbf{Z}/2)$ or $\hat{\mathbf{Z}}_2$, depending on whether or not $\sigma_{-1} \in \mathcal{G}(\mathbf{Q}_2^c/K)$. If $\sigma_{-1} \notin \mathcal{G}(\mathbf{Q}_2^c/K)$, then it follows that the inertia group of $\mathbf{Q}_2(\zeta)/K$ is always cyclic. Suppose $\sigma_{-1} \in \mathcal{G}(\mathbf{Q}_2^c/K)$. Then $\mathcal{G}(\mathbf{Q}_2^c/K_{nr}) = \hat{\mathbf{Z}}_2 \times \mathbf{Z}/2$ where the first factor is topologically generated by $\sigma_5^{2^k}$ for some $k \geq 0$ and $\mathbf{Z}/2$ is generated by σ_{-1} . If we choose ζ to have order divisible by a power of 2 large enough so that $\sigma_5^{2^k}(\zeta) \neq \zeta$, then it is clear that the inertia subgroup of $\mathbf{Q}_2(\zeta)/K$ is not cyclic. Thus the inertia group of $\mathbf{Q}_2(\zeta)/K$ is non-cyclic iff $\sigma_{-1} \in \mathcal{G}(\mathbf{Q}_2^c/K)$, and so Yamada's criterion is equivalent to mine.

2. U. Fontaine, [F], Cor. 2', p. 138. The result is: $S(K)$ is non-trivial iff $\varepsilon_4 \notin K$. This is easily seen to be inequivalent to the other formulations. As an example, let K be the subfield of $\mathbf{Q}_2(\varepsilon_{16})$ fixed by the automorphism $\sigma_{-1}\sigma_5^2$. Then $\varepsilon_4 \notin K$ and $\sigma_{-1} \notin \mathcal{G}(\mathbf{Q}_2^c/K)$.

3. G. J. Janusz, [J], p. 543. Let h be the smallest integer ≥ 2 such that there is an odd integer $c \geq 1$ with the property that $\mathbf{Q}_2(\varepsilon_{2^h}, \varepsilon_c)$ contains K . Then Janusz' theorem is the following:

$S(K)$ is non-trivial iff there is an odd integer n with the following properties:

- (i) $K(\varepsilon_4)/K$ is ramified.
- (ii) $K(\varepsilon_{4n}) = \mathbf{Q}_2(\varepsilon_{2^h}, \varepsilon_n)$.
- (iii) $(K(\varepsilon_n):K) = 2^r w$, where w is odd and $r \geq 1$.
- (iv) The automorphism of order 2 in $\mathcal{G}(K(\varepsilon_{4n})/K(\varepsilon_n))$ carries ε_{2^h} to $\varepsilon_{2^h}^{-1}$.
- (v) If $r \leq h-1$, then any root of unity in $K(\varepsilon_{4n})$ whose order divides 2^{h-r+1} already lies in $K(\varepsilon_4)$.

It can be shown that the conditions (iii) and (v) can be omitted. Indeed suppose that we are given an odd integer n such that (i), (ii), and (iv) are

satisfied. Let the residue class field of $K(\varepsilon_n)$ have 2^k elements. Set $n' = (2^k)^{2^h} - 1$. Then $n \mid n'$, n' is odd, and $K(\varepsilon_{n'})/K(\varepsilon_n)$ is unramified of degree 2^h . Consider the conditions (i)-(v) with n' instead of n . Then (i) is unchanged, (ii) holds because $n \mid n'$, (iii) holds trivially and (v) holds vacuously because $2^h \mid (K(\varepsilon_{n'}) : K)$. Finally $K(\varepsilon_{n'}) \cap K(\varepsilon_4) = K$ since one is ramified and the other is not, so the non-trivial automorphism of $K(\varepsilon_{4n})/K(\varepsilon_n)$ is the restriction of that of $K(\varepsilon_{4n'})/K(\varepsilon_{n'})$, so (iv) holds also for n' .

We can deduce from this abbreviated form of Janusz' theorem that it is equivalent to Yamada's. Suppose that Janusz' conditions are satisfied, and consider the extension $\mathbf{Q}_2(\varepsilon_{2^{h+1}}, \varepsilon_n)/K$. The inertia subgroup of its Galois group is $\mathcal{g} = \mathcal{G}(\mathbf{Q}_2(\varepsilon_{2^{h+1}}, \varepsilon_n)/K(\varepsilon_n))$, a group of order 4. Suppose that ρ is an extension of the non-trivial automorphism of $\mathbf{Q}_2(\varepsilon_{2^h}, \varepsilon_n)/K(\varepsilon_n)$ to $\mathbf{Q}_2(\varepsilon_{2^{h+1}}, \varepsilon_n)$, so $\rho \in \mathcal{g}$. By condition (iv), there is an integer $a \equiv -1 \pmod{2^h}$ such that $\rho(\varepsilon_{2^{h+1}}) = \varepsilon_{2^{h+1}}^a$. It follows that ρ^2 is the identity. Thus $\mathcal{g}$ is non-cyclic. Conversely suppose that there is an extension $\mathbf{Q}_2(\zeta)/K$ whose inertia subgroup $\mathcal{g}$ is non-cyclic. As we saw in 1., this means that σ_{-1} is in the Galois group of $\mathbf{Q}_2^c/K$ and so its restriction (which we also call σ_{-1}) is in $\mathcal{G}(\mathbf{Q}_2(\varepsilon_{2^h}, \varepsilon_c)/K)$ and is non-trivial. Its fixed field contains $K(\varepsilon_c)$; by Lemma 3.3 of [J], $K(\varepsilon_c, \varepsilon_4) = \mathbf{Q}_2(\varepsilon_{2^h}, \varepsilon_c)$ and so the fixed field is *exactly* $K(\varepsilon_c)$. Thus both (iv) and (ii) are also fulfilled. (i) holds by Lemma 1.

4. F. Lorenz, [L], p. 463. His condition for *non-triviality of $S(K)$* is that -1 is a norm in the extension $K/\mathbf{Q}_2$. The norm residue symbol in the extension $\mathbf{Q}_2^c/\mathbf{Q}_2$ sends -1 to $\sigma_{-1} \in \mathcal{G}(\mathbf{Q}_2^c/\mathbf{Q}_2)$. Thus it follows from [S], pp. 204-205, that -1 is a norm in $K/\mathbf{Q}_2$ iff $\sigma_{-1} \in \mathcal{G}(\mathbf{Q}_2^c/K)$.

REFERENCES

- [C-F] CASSELS, J. W. S. and A. FROHLICH. *Algebraic Number Theory*. Thompson Book Co. Inc., Washington (1967).
- [F] FONTAINE, J.-M. Sur la décomposition des algèbres de groupes. *Ann. Sc. Ec. Norm. Sup.* 49 (1971), 121-180.
- [H] HASSE, H. *Zahlentheorie*, 2nd ed. Akademie-Verlag, Berlin, 1963.
- [I] ISAACS, I. M. *Character Theory of Finite Groups*. Academic Press, New York (1976).