# §2. The minimal B/-embeddings


Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.09.2024**

To describe a $B/\Gamma$-embedding with underlying variety $X$, we must give a homomorphism $B \to \operatorname{Aut} X$ such that $X$ has an open orbit $B$-isomorphic to $B/\Gamma$. Two such homomorphisms give rise to equivalent embeddings if and only if they are conjugate.

In the following section we will use the information given here to study the possible $B/\Gamma$-embeddings into $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$, and $\mathbf{F}_n$, $n \geqslant 1$.

## § 2. The minimal $B/\Gamma$-embeddings

Theorem 2.1. *Let* $\Gamma$ *be a finite subgroup of* $B$, *and let* $X$ *be the projective plane* $\mathbf{P}^2$ *or a rational ruled surface* $\mathbf{F}_n$ *(with* $n \geqslant 0$, *where* $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$*).*

(i) *The number* $\operatorname{emb}(X)$ *of equivalence classes of* $B/\Gamma$*-embeddings into* $X$ with at least two fixed points *is*

$$\operatorname{emb}(\mathbf{P}^2) = 2\,, \quad \operatorname{emb}(\mathbf{P}^1 \times \mathbf{P}^1) = 1\,, \quad \textit{and} \quad \operatorname{emb}(\mathbf{F}_n) = n + 3,\ n \geqslant 1\,.$$

*We call these the "ordinary" embeddings.*

(ii) *Moreover, for any such surface* $X$, *there is exactly one subgroup* $\Gamma$ *and an "exceptional"* $B/\Gamma$*-embedding into* $X$ with only one fixed point *(up to equivalence), and the corresponding order* $\operatorname{ord}(X)$ *of this group* $\Gamma$ *is*

$$\operatorname{ord}(\mathbf{P}^2) = 4\,, \quad \operatorname{ord}(\mathbf{P}^1 \times \mathbf{P}^1) = 2\,, \quad \textit{and} \quad \operatorname{ord}(\mathbf{F}_n) = 2(n+1),\ n \geqslant 1\,.$$

(iii) *The complement to the open orbit consists of two (for* $\mathbf{P}^2$*) resp. three (for the* $\mathbf{F}_n$*) smooth rational curves, intersecting transversely, except in the "exceptional" case with* $X = \mathbf{P}^2$, *in which case the two curves are tangent.*

(In this theorem we include the case $\mathbf{F}_1$ even though it is not minimal.)

To be more precise, we indicate the form of the complement $Z$ to the open orbit in each case. Also to distinguish the embeddings where $Z$ has the same form, we indicate how the action of $B$ differs on $Z$. Let $U$ be the unipotent radical of $B$ and $T$ be a maximal torus. (That is, $U$ is the subgroup of elements of $B$ where both eigenvalues are 1, and $T$ can be chosen to be the subgroup of diagonal elements.) Then $B$ is $T \ltimes U$, and the characters of $B$ are the characters of $T$. We denote the character group of $B$ by $\{\alpha^n : n \in \mathbf{Z}\}$.

Denote by $c$ the order of the group $\Gamma$.

*Embeddings into* $\mathbf{P}^2$:

(i) "Ordinary" embeddings: We find that for each $\Gamma$ there are two embeddings where $Z = L_1 \cup L_2$ and $L_1$ and $L_2$ are lines in $\mathbf{P}^2$. The group $B$ acts on $L_1$ in the standard manner and on $L_2$ by the character $\alpha^{2+c}$ or $\alpha^{2-c}$. There are two fixed points except in one embedding for the case $c = 2$, where $L_2$ is a line of fixed points. See Fig. 1a.

(ii) The "exceptional" embedding: If $c = 4$, we also find an embedding where $Z = L_1 \cup C$ and $C$ is a smooth conic which is tangent to $L_1$ at the unique fixed point. See Fig. 1b.

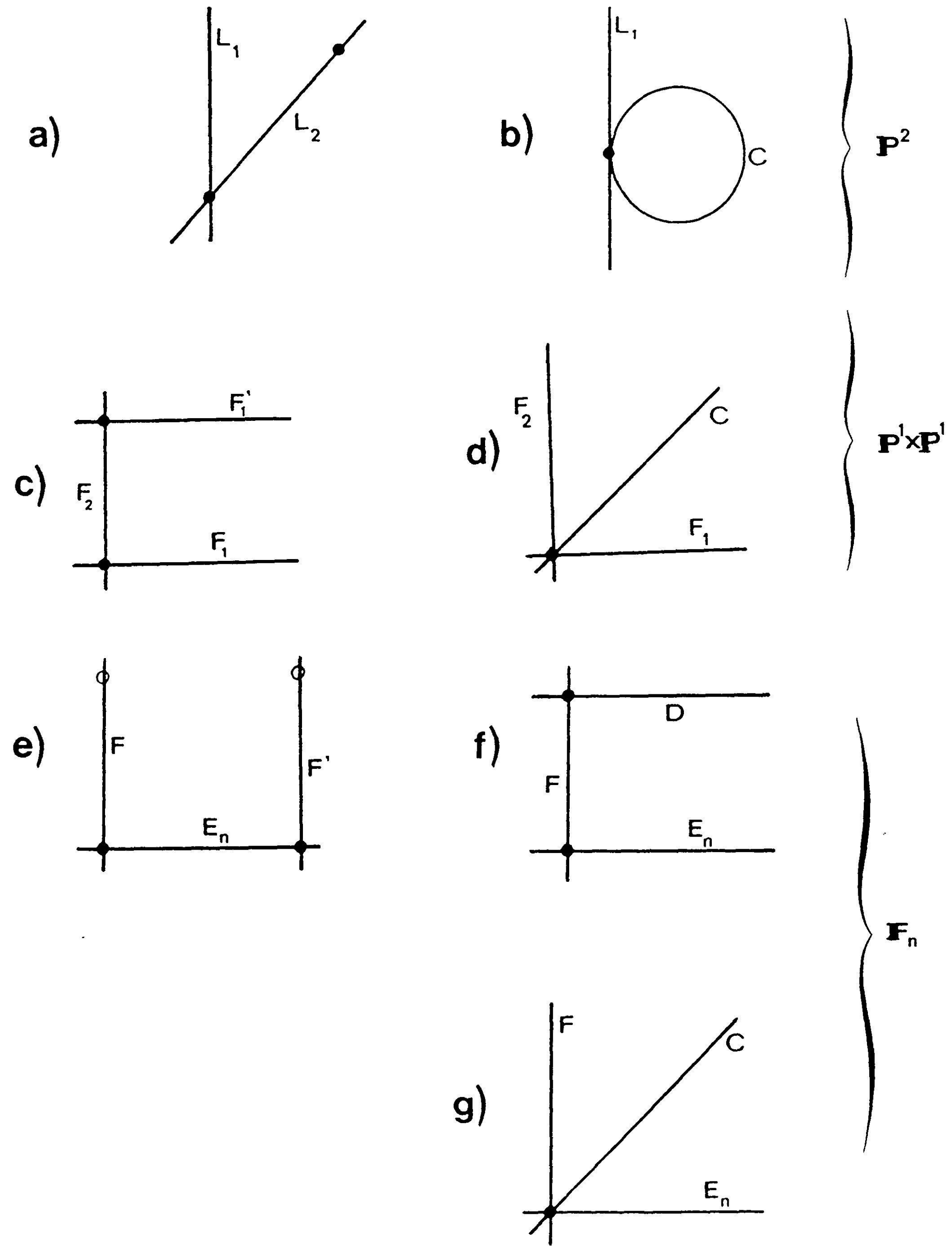


FIGURE 1.

*Embeddings into* $\mathbf{P}^1 \times \mathbf{P}^1$:

In this case, $Z$ is always the union of three curves. Let $p_i: \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$, $i = 1, 2$ be the two projections.

(i) "Ordinary" embeddings: For each $\Gamma$ there is an embedding where $Z = F_1 \cup F'_1 \cup F_2$ and $F_1, F'_1$ are fibres of $p_1$ and $F_2$ is a fibre of $p_2$. There are two fixed points. See Fig. 1c.

(ii) The "exceptional" embedding: Also, if $c = 2$, we find another embedding into $\mathbf{P}^1 \times \mathbf{P}^1$ where $Z = F_1 \cup F_2 \cup C$, and $C$ is a section of $p_1$ and $p_2$ which intersects $F_1$ and $F_2$ transversely in the unique fixed point. See Fig. 1d.

*Embeddings into* $\mathbf{F}_n, n \geqslant 1$:

Again $Z$ is is always the union of three curves. Let $\pi_n: \mathbf{F}_n \to \mathbf{P}^1$ be the unique ruling of $\mathbf{F}_n$, and let $E_n$ be the irreducible curve of $\mathbf{F}_n$ with self-intersection $-n$.

(i) "Ordinary" embeddings: For each $\Gamma$ we find $n+1$ cases where $Z = E_n \cup F \cup F'$ and $F$ and $F'$ are fibres of $\pi_n$. The torus $T$ acts on $F$ by the character $\alpha^{cp+2}$ and on $F'$ by the character $\alpha^{-c(n-p)+2}$, $p = 0, \ldots, n$. There are either 3 or 4 fixed points (depending on the action of $U$ on $F$ and $F'$), or, if $T$ acts trivially on $F'$, then $F'$ is a curve of fixed points. See Fig. 1e.

There are also two other embeddings in $\mathbf{F}_n$ for each $\Gamma$ where $Z = F \cup E_n \cup D$ and $F$ is a fibre as before and $D$ is a section of $\pi_n$ which does not intersect $E_n$. The group $B$ acts on $F$ by the character $\alpha^{2n \pm c}$. There are two fixed points except in one of the embeddings in the case where $c = 2n$, in which case $F$ consists entirely of fixed points. See Fig. 1f.

(ii) "Exceptional" embeddings: Also if $c = 2(n+1)$, there is one more embedding where $Z = E_n \cup F \cup C$ and $C$ is a section which intersects $E_n$ and $F$ transversely in the unique fixed point. See Fig. 1g. This embedding is obtained as follows. Consider the embedding into $\mathbf{F}_{n+1}$ of the previous type where the fibre $F$ consists of fixed points. Blow up a point of $F$ which is not on $E_{n+1}$ or $D$ and contract the strict transform of $F$. This gives the required embedding into $\mathbf{F}_n$.

The explicit matrix representations of the different $B$-actions are given in the proof of the theorem.

*Proof of the Theorem.* Throughout the proof we denote the order of the group $\Gamma$ by $c$.

Recall that to give an embedding of $B/\Gamma$ into a variety $X$, we must find a homomorphism $\varphi: B \to \operatorname{Aut} X$ such that under the induced action of $B$ on $X$, there is an open orbit isomorphic to $B/\Gamma$. Two such embeddings are equivalent if and only if the homomorphisms are conjugate.

We have $B = \left\{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha \in k^* \text{ and } \beta \in k\right\}$, $U = \left\{\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \beta \in k\right\}$, and set $T = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha \in k^*\right\}$.

We consider separately the embeddings into $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{F}_n$, $n \geqslant 1$.

*Embeddings into* $\mathbf{P}^2$*:*

If $B$ acts on $\mathbf{P}^2$, it has a fixed point $o$ since $\mathbf{P}^2$ is complete and $B$ is solvable (see e.g. [Bor], p. 242). Also $B$ acts on the linear system $S = \{$lines of $\mathbf{P}^2$ passing through $o\}$. Since we have $S \cong \mathbf{P}^1$, $B$ stabilizes one such line, which we call $L$. We can choose homogeneous coordinates $(z_0 : z_1 : z_2)$ of $\mathbf{P}^2$ such that $o = (1:0:0)$ and $L = (z_0 : z_1 : 0)$; thus $\varphi(B) \subset PGL(3)$ is upper triangular.

CASE 1. *U acts trivially on L.*

Then there is another point $o' \in L$ fixed by $B$. By choosing an appropriate basis, we can assume that for $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in U$ we have

$$\varphi\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \in PGL(3) .$$

The brackets indicate the class of the matrix in $PGL(3)$. All the lines passing through $o'$ are stable by $U$. By a change of basis we can also assume that $\varphi(T)$ is diagonal. Then for $\varphi$ to be a homomorphism, it is necessary that

$$\varphi\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{bmatrix} \alpha^m & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha^{-1} \end{bmatrix} \in PGL(3) , \quad m \in \mathbf{Z} .$$

For $m = -1 \pm c$, this gives two embeddings of $B/\Gamma$ with $|\Gamma| = c$. The group $B$ acts on $L$ by the character $\alpha^{2 \pm c}$. There is another stable line

$\{(0:z_1:z_2) \mid z_i \in k\}$ on which $B$ acts in the standard manner. This gives the two "ordinary" $B/\Gamma$-embeddings mentioned earlier for $\mathbf{P}^2$.

CASE 2. *U acts non-trivially on L.*

(i) $U$ acts trivially on the linear system $S$.

Then $B$ stabilizes another line $L'$ passing through $o$. Since we have that $\mathbf{P}^2 - \{L \cup L'\} \cong k \times k^* \cong B/\Gamma$, and since $k \times k^*$ contains no proper open subvariety isomorphic to itself, we must have that the complement to the open orbit is $Z = L \cup L'$. We will show that $U$ acts trivially on $L'$. Indeed, let $x \in L' \backslash L$ and $D$ be a line of $\mathbf{P}^2$ passing through $x$ but not $o$, and let $u \in U$, $u \neq e$; then $uD \cap D$ is a point fixed by $u$ since $U$ acts trivially on $S$; therefore it must belong to $Z$, but it is not in $L$; thus it is in $L'$, hence it is $x$. So by exchanging $L$ and $L'$, we are in Case 1.

(ii) $U$ acts non-trivially on the linear system $S$.

Then $T$ stabilizes a line $L'$ in $S - L$.

Fix $u \in U$, $u \neq e$. We can choose a basis such that $\varphi(u)$ is in Jordan normal form $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Now by a change of basis we can assume

$$\varphi\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 2\beta & \beta^2 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \in PGL(3) .$$

Let $S'$ be the linear system of conics passing through the point $o$. Now $B$ acts on $S'$, and one can easily check that ${}^{U}S'$, the set of conics stable by $U$ is isomorphic to $\mathbf{P}^1$. In fact it is the set of conics of the form

$$\{(z_0:z_1:z_2) \mid a(z_0z_2 - z_1^2) + bz_2^2 = 0\}, \quad (a:b) \in \mathbf{P}^1 .$$

Also $T$ acts on ${}^{U}S'$; it must leave two conics invariant: the double line $L = \{(z_0:z_1:0)\}$ and a non-degenerate conic $C$. Since $\mathbf{P}^2 - \{L \cup C\}$ is isomorphic to $k \times k^*$, the complement to the open orbit is $L \cup C$. By a change of basis one can choose

$$C = \{(z_0:z_1:z_2) \mid z_0z_2 - z_1^2 = 0\} \quad \text{and} \quad L' = \{(z_0:0:z_1)\} .$$

By checking the action of $T$ on $\mathbf{P}^2 - L$, one finds there is just one possibility which yields:

$$\varphi\begin{pmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{pmatrix} = \begin{bmatrix}\alpha^2 & 2\alpha\beta & \beta^2\\ 0 & 1 & \alpha^{-1}\beta\\ 0 & 0 & \alpha^{-2}\end{bmatrix} \in PGL(3)\ .$$

(So $\varphi$ is obtained from the irreducible representation of $SL(2)$ of dimension 3.) This homomorphism gives rise to a $B/\Gamma$-embedding for $c = 4$. Note that there is exactly one fixed point: $(1:0:0)$. This is the "exceptional" embedding.

*Embeddings into* $\mathbf{P}^1 \times \mathbf{P}^1$:

The two projections $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ give the two different rulings of $\mathbf{P}^1 \times \mathbf{P}^1$. Any automorphism of $\mathbf{P}^1 \times \mathbf{P}^1$ either leaves the two rulings invariant or exchanges them. In other words,

$$\mathrm{Aut}\,(\mathbf{P}^1 \times \mathbf{P}^1) = \big(PGL(2) \times PGL(2)\big) \rtimes \mathbf{Z}/2\mathbf{Z}\,.$$

Since $B$ is connected, the image of $\varphi(B) \subset \mathrm{Aut}\,(\mathbf{P}^1 \times \mathbf{P}^1)$ is connected; thus we consider homomorphisms $\varphi: B \to PGL(2) \times PGL(2)$. Up to conjugation, the only homomorphisms of $B$ to $PGL(2)$ are

$$\begin{pmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{pmatrix} \to \begin{bmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{bmatrix} \in PGL(2)$$

or
$$\begin{pmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{pmatrix} \to \begin{bmatrix}\alpha^m & 0\\ 0 & 1\end{bmatrix} \in PGL(2)\,,\ m = 0, 1, 2, \ldots\ .$$

To obtain an embedding, $U$ cannot act trivially on $\mathbf{P}^1 \times \mathbf{P}^1$. So the possibilities (up to conjugation) are

$$\varphi\begin{pmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{pmatrix} = \left\{\begin{bmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{bmatrix}, \begin{bmatrix}\alpha^m & 0\\ 0 & 1\end{bmatrix}\right\} \in \mathrm{Aut}\,(\mathbf{P}^1 \times \mathbf{P}^1)\,,\ m = 1, 2, 3, \ldots$$

or

$$\varphi\begin{pmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{pmatrix} = \left\{\begin{bmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{bmatrix}, \begin{bmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{bmatrix}\right\} \in \mathrm{Aut}\,(\mathbf{P}^1 \times \mathbf{P}^1)\,.$$

In the first case, we get an "ordinary" embedding of $B/\Gamma$ with $c = m$ with two fixed points. The second induces a $B/\Gamma$-embedding with $c = 2$, and the complement to the open orbit consists of three curves isomorphic to $\mathbf{P}^1$ all intersecting transversely in the unique fixed point. This is the "exceptional" embedding.

*Embeddings into* $\mathbf{F}_n, n \geqslant 1$:

Remember from section 1 that we can consider $\mathbf{F}_n$ as the union of $E_n$ and the total space of the line bundle $\mathcal{O}_{\mathbf{P}^1}(n)$. Suppose we have a homomorphism $\varphi: B \to \operatorname{Aut} \mathbf{F}_n$ which gives rise to a $B/\Gamma$-embedding. Since $\operatorname{Aut} \mathbf{F}_n$ stabilizes $E_n$, we know that $B$ fixes $E_n$. We consider three cases.

CASE 1. *$U$ acts trivially on $E_n$.*

We will find $n + 1$ inequivalent "ordinary" embeddings of this type for each $\Gamma$.

In this case, consider the action of $T$ on $E_n$. It cannot act trivially (because then each $B$-orbit would be contained in a fibre of $\pi_n: \mathbf{F}_n \to \mathbf{P}^1$) and has therefore exactly two fixed points, $x$ and $y$. By possibly exchanging $x$ and $y$, we can assume that $T$ acts by a character $\alpha^m, m > 0$ on $E_n \cong \mathbf{P}^1$ (i.e. for $z \in E_n - \{x, y\}$, we choose $x = \lim\limits_{t \to 0} tz$ and $y = \lim\limits_{t \to \infty} tz, t \in T$).

The fibres $F_x$ and $F_y$ of $x$ and $y$, respectively, are stable by $B$. Let $Z$ be the complement of the open orbit in $\mathbf{F}_n$. Then we have $E_n \cup F_x \cup F_y \subset Z$. Since we know that $\mathbf{F}_n - \{E_n \cup F_x \cup F_y\} \cong k \times k^* \cong B/\Gamma$, and since, as noted earlier, $k \times k^*$ contains no proper open subvariety isomorphic to itself, we must have $Z = E_n \cup F_x \cup F_y$.

Now by Lemma 1.3, we have $T \hookrightarrow B \to \operatorname{Aut} \mathbf{F}_n \to \operatorname{Aff}(H^0(\mathbf{P}^1, \mathcal{O}(n))$. Since $T$ is reductive, $T$ must fix a section $D$ of $\mathcal{O}(n)$.

We also have that $U$ acts on the space $H^0(\mathbf{P}^1, \mathcal{O}(n))$. Consider the orbit $UD$. First note that $UD \cong k$ (we could not have $UD = D$, because then $D$ would be in the complement of the open orbit). Now let $u \in U$, $u \neq e$; then I claim that $uD \cap D \subset \{x', y'\}$, where $x' = F_x \cap D$ and $y' = F_y \cap D$. To see this, note that since $U$ acts trivially on $E_n$, it stabilizes the fibres of $\pi_n$. Thus if $z$ belongs to $uD \cap D$, then $u$ belongs to the isotropy group of $z$, and therefore $z$ must be in $Z$. The intersection number $uD \cdot D$ is $n$; so $UD \subset D \cup \bigcup\limits_{p=0}^{n} A_p$, where $A_p$ is the set of sections $D'$ of $\mathcal{O}(n)$ such that $D \cap D' = px' + (n-p)y'$ counted with multiplicity. Now $D \cup A_p$ is isomorphic to $k$, $p = 0, ..., n$; so $UD = D \cup A_p$ for some $p = 0, ..., n$. We call $p$ the *contact index* of the embedding. See Fig. 2.

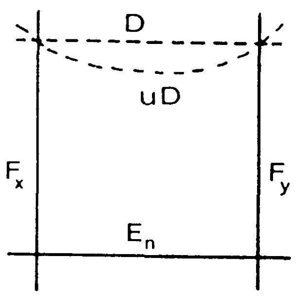


FIGURE 2.

LEMMA 2.2. *Up to equivalence, there is at most one $B/\Gamma$-embedding into* $\mathbf{F}_n$ *of a given contact index* $p$, *with* $p = 0, \ldots, n$. *Also, for such an embedding* $B$ *acts on* $E_n$ *by the character* $\alpha^c$, *where* $c$ *is the order of* $\Gamma$.

*Proof.* Suppose we have two $B/\Gamma$-embeddings into $\mathbf{F}_n$ with the same contact index $p$. Fix $u \in U$, $u \neq e$. For the first (resp. second) action denote by $x, y$ (resp. $\tilde{x}, \tilde{y}$) the fixed points in $E_n$ and $D$ (resp. $\tilde{D}$) the section fixed by $T$. Set $D_u := uD$ (resp. $\tilde{D}_u := u\tilde{D}$).

Remember from section 1 we know that there is an exact sequence

$$1 \to k^* \ltimes H^0(\mathbf{P}^1, \mathcal{O}(n)) \to \operatorname{Aut} \mathbf{F}_n \to PGL(2) \to 1 .$$

Since $PGL(2)$ acts doubly transitively on $\mathbf{P}^1$, we can conjugate by an automorphism of $\mathbf{F}_n$ which sends $x$ to $\tilde{x}$ and $y$ to $\tilde{y}$; thus we can assume $x = \tilde{x}$ and $y = \tilde{y}$. Then by conjugating by an element of $H^0(\mathbf{P}^1, \mathcal{O}(n))$, which translates the sections, we can assume $D = \tilde{D}$. Finally, since the two embeddings have the same contact index, by conjugating by an automorphism that fixes the fibres and which is a homothety centered at $D$, we can assume $D_u = \tilde{D}_u$.

Now I claim that for a fixed $\Gamma$, there is at most one possible action of $B$ on $\mathbf{F}_n$ which induces a $B/\Gamma$-embedding with the quadruple $\{x, y, D, D_u\}$. Indeed $U$ acts by translation on each of the fibres of $\mathcal{O}(n)$; so $D$ and $D_u$ determine how $U$ must act. Now check the action of $T$ on $D$, which is the same as its action on $E_n$. Choose $z \in D$ in the open orbit. The order of the isotropy group $B_z$ is $c$, the order of $\Gamma$, and $B_z \subset T$. So $T$ acts on $D$ by a character $\alpha^{\pm c}$. Since we chose $x$ and $y$ such that the action of $T$ on $E_n$ is given by a positive character, we must have that $T$ acts on $D$ by the character $\alpha^c$. This proves the second statement of the lemma. Now let $v$ be an element of the open orbit and $t \in T$. Choose $u \in U$ such that $(t^{-1}ut)v = v' \in D$. Then $tv = u^{-1}tv'$. So this fixes the action of $T$ on the open orbit, which is dense in $\mathbf{F}_n$. So the claim is true, and this finishes the proof of the lemma. □

By this lemma, we have at most $n + 1$ inequivalent embeddings of this type for each $\Gamma$. Now we must show that these actually exist.

LEMMA 2.3. *Let* $n$ *be a positive integer and* $p$ *be an integer such that* $0 \leqslant p \leqslant n$. *Then for each finite* $\Gamma \subset B$, *there exists a* $B/\Gamma$*-embedding into* $\mathbf{F}_n$ *with contact index* $p$.

*Proof.* Let $X_n$ be the surface obtained by contracting $E_n$ in $\mathbf{F}_n$ as explained in section 1. Suppose we have an embedding of $B/\Gamma$ into $X_n$

which fixes the vertex of the cone (if $n > 1$, this condition is always satisfied, because this point is singular). Then by blowing up the vertex, we obtain an embedding into $\mathbf{F}_n$.

For each $p$ with $0 \leqslant p \leqslant n$, we will exhibit an action of $B$ on $X_n$ which induces a $B/\Gamma$-embedding with contact index $p$. To do this we give a linear action of $B$ on $k^{n+2}$ which induces an action of $B$ on $\mathbf{P}^{n+1}$ stabilizing $X_n$ and its vertex.

$B$ acts on $k^2$ in the standard way:

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \alpha s + \beta t \\ \alpha^{-1} t \end{pmatrix}.$$

Also for $i \in \mathbf{Z}$, we denote by $(k, \alpha^i)$ the vector space $k$ with the action of $B$ by the character $\alpha^i$:

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} z = \alpha^i z .$$

Consider the $B$-module

$$k^2 \otimes (k, \alpha^{cp+1}) \oplus \bigoplus_{\substack{j=0 \\ j \neq p}}^{n} (k, \alpha^{cj}), \quad p = 0, \ldots, n .$$

We have $B \to PGL(n+2) = \operatorname{Aut} \mathbf{P}^{n+1}$ by

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \mapsto \begin{bmatrix} \alpha^{cp+2} & \alpha^{cp+1}\beta & 0 & \cdots & & & & 0 \\ 0 & \alpha^{cp} & & & & & & \vdots \\ \vdots & & 1 & & & 0 & & \\ & & & \alpha^{c} & & & & \\ & & & & \ddots & & & \vdots \\ & 0 & & & & \widehat{\alpha^{cp}} & & \\ \vdots & & & & & & \ddots & \\ 0 & & & \cdots & & & 0 & \alpha^{cn} \end{bmatrix}$$

We change the basis so that the image of $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$ is

$$\begin{bmatrix} \alpha^{cp+2} & 0 & \dots 0 & \alpha^{cp+1} & \beta & 0 & \dots & 0 \\ 0 & 1 & & & & & & \\ \cdot & & \alpha^{c} & & & & 0 & \cdot \\ \cdot & & & \ddots & & & & \cdot \\ \cdot & 0 & & & \alpha^{cp} & & & \cdot \\ & & & & & & \ddots & 0 \\ 0 & & & \dots & & & 0 & \alpha^{cn} \end{bmatrix}$$

Let $X_n$ be as given in section 1. Clearly $X_n$ and the vertex of the cone $(1:0:\ldots:0)$ are fixed by this action. In $X_n$ all the "fibres" are stable by $U$, and the two "fibres" $F_x = \{(z_0:z_1:0:\ldots:0)\}$ and $F_y = \{(z_0:0:\ldots:0:z_{n+1})\}$ are stable by $B$. It is easy to check that the isotropy group of $(0:1:\ldots:1)$ is the finite subgroup of $T$ of order $c$. So this induces an embedding of $B/\Gamma$ into $X_n$ which by blowing up the vertex gives a $B/\Gamma$-embedding into $\mathbf{F}_n$ where $U$ acts trivially on $E_n$.

Let $D = \{(0:s^n:s^{n-1}t:\ldots:t^n)\} \subset X_n$. Then $D$ is a "section" stable by $T$. Fix $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in U$. Then $uD = \{(s^{n-p}t^p:s^n:s^{n-1}t:\ldots:t^n)\} \subset X_n$. We check the multiplicity of the intersection of $D$ and $uD$ at $x' = (0:1:0:\ldots:0)$. The local ring of $x'$ in $X_n$ is $k[z_0, t]_{(t, z_0)}$, and the local equation of $D$ (resp. $uD$) is $z_0 = 0$ (resp. $z_0 = t^p$); thus this multiplicity is $p$, and the contact index of the embedding is $p$. This finishes the proof of the lemma. □

*Remark.* By checking the induced torus actions on the fibres $F_x$ and $F_y$, one finds the results about the structure of the action stated after Theorem 2.1.

CASE 2. *$U$ acts non-trivially on $E_n$ and $B$ fixes a section $D$ of $\mathcal{O}(n)$.*

We will find two "ordinary" embeddings of this type for each $\Gamma$.

In this case, $U$ has one fixed point $x$ on $E_n$. Then $T$ must also fix $x$, and it also fixes another point $y \in E_n$. As before, we call $Z$ the complement to the open orbit. Then we have $Z = E_n \cup D \cup F_x$, where $F_x$ is the fibre of $\pi_n$ containing $x$. Now look at the action of $T$ on $F_y$, the fibre of $y$. Choose $z \in F_y$ in the open orbit. Then the order of the isotropy group $B_z$

is $c$, the order of $\Gamma$, and $B_z \subset T$. So $T$ acts on $F_y$ by the character $\alpha^{\pm c}$. For each such embedding, call this character the *sign* of the embedding. See Fig. 3.

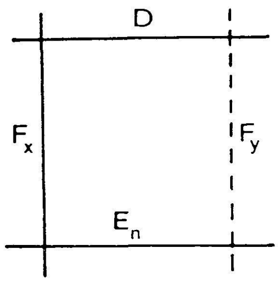


FIGURE 3.

LEMMA 2.4. *Up to equivalence, there is at most one $B/\Gamma$-embedding into* $\mathbf{F}_n$ *with a given sign* $\sigma = \alpha^{\pm c}$.

*Proof.* Suppose we had two actions of $B$ on $\mathbf{F}_n$ which yield two $B/\Gamma$-embeddings with the same sign $\sigma$. For the first (resp. second) action, let $\psi$ (resp. $\tilde{\psi}$): $B \times E_n \to E_n$ be the induced action on $E_n$ and $D$ (resp. $\tilde{D}$) be the section of $\mathcal{O}(n)$ fixed by $B$.

Up to conjugacy there is only one action of $B$ on $E_n \cong \mathbf{P}^1$ for which $U$ acts non-trivially. So we can assume $\psi = \tilde{\psi}$. By conjugating by an appropriate automorphism of $\mathbf{F}_n$ which fixes the fibres and translates the sections, we can assume $D = \tilde{D}$.

Now I claim there is at most one action of $B$ on $\mathbf{F}_n$ which yields a $B/\Gamma$-embedding with the triple $\{\psi, D, \sigma\}$. To see this, consider first the action of $U$ on $\mathbf{F}_n$. Now $x$ is the fixed point of $E_n$, and $F_x$ is its fibre. Let $S$ be the set of sections of $\mathcal{O}(n)$ which are not $D$ and intersect $D$ with multiplicity $n$ at the fixed point $x' = F_x \cap D$. This set is isomorphic to $k^*$ (by the map $D' \to D' \cap F_y$) and is stable by $B$, so $U$ acts trivially on $S$. Since the action of $U$ on $D' \in S$ is identical to its action on $E_n$, the action of $U$ on $\mathbf{F}_n$ is determined by $\psi$ and $D$. As for the action of $T$, remember that $T$ stabilizes the set $S$. The action on this set is equivalent to its action on $F_y$, the fibre of the point of $E_n$ fixed by $T$ and not fixed by $U$. This action is given by $\sigma$. So $\{\psi, D, \sigma\}$ determines the action of $T$ on $\mathbf{F}_n$. This proves the claim. □

From this lemma, we see that for each $\Gamma$, there is at most two $B/\Gamma$-embeddings of this type. Now we must show that these embeddings actually exist.

LEMMA 2.5. *Let* $\Gamma$ *be a finite subgroup of* $B$ *of order* $c$ *and* $\sigma$ *be* $\alpha^{\pm c}$. *Then there exists a* $B/\Gamma$*-embedding into* $\mathbf{F}_n$ *with sign* $\sigma$.

*Proof.* We use the same notation as in Lemma 2.3. Consider the $B$-module

$$(k, \alpha^{-n \pm c}) \oplus S^n(k^2)$$

where $S^n(k^2)$ is the vector space of homogeneous polynomials of degree $n$ over $k$ with two variables, and the action of $B$ on $S^n(k^2)$ is induced from the natural action on $k^2$ of $B$ as a subgroup of $SL(2)$. We have $B \to PGL(n+2)$ by

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \mapsto \begin{bmatrix} \alpha^{-n \pm c} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \rho_n \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} & \\ 0 & & & \end{bmatrix}$$

where $\rho_n$ is the $(n+1)$-dimensional irreducible matrix representation of $SL(2, k)$ corresponding to the basis $\{\binom{n}{i} x^i y^{n-i}\}_{i=0,\dots,n}$ of $S^n(k^2)$.

As in Lemma 2.3, let $X_n = \{(z_0 : s^n : s^{n-1}t : \dots : t^n) \mid z_0, s, t \in k\} \subset \mathbf{P}^{n+1}$. Then $X_n$ and its vertex $(1:0:\dots:0)$ are fixed by the action above. In $X_n$ the "section" $\{(0 : s^n : \dots : t^n)\}$ and the "fibre" $\{(z_0 : z_1 : 0 : \dots : 0)\}$ are stable. The other "fibres" are not stable by $U$. The isotropy group of $(1:0:\dots:0:1)$ is the finite subgroup of $T$ of order $c$. So this action gives an embedding of $B/\Gamma$ into $X_n$ which by blowing up the vertex gives an embedding into $\mathbf{F}_n$ where $U$ acts non-trivially on $E_n$ and $B$ fixes a section.

The "fibre" $\{(z_0 : 0 : \dots : 0 : z_{n+1})\}$ is stable by $T$ and not by $U$. Also $T$ acts on this fibre" by the character $\alpha^{\pm c}$, so the sign of the embedding is $\alpha^{\pm c}$. This proves the lemma. □

*Remark.* The group $B$ acts on the fixed fibre of the $B/\Gamma$-embedding with sign $\alpha^{\pm c}$ by the character $\alpha^{2n \mp c}$. In particular, for each $n$, there is exactly one embedding of this type with $c = 2n$ where $B$ acts trivially on the fixed fibre. We will use this remark for the following case.

CASE 3. *$U$ acts non-trivially on $E_n$ and $B$ does not fix any section of $\mathcal{O}(n)$.*

For each $n$, we find one such case where $c = 2(n+1)$. These are the "exceptional" embeddings.

As in the previous case, $B$ fixes one element $x \in E_n$. So $Z$, the complement to the open orbit, contains $E_n$ and $F_x$, the fibre of $x$. Now $\mathbf{F}_n - \{E_n \cup F_x\}$ is isomorphic to $k \times k$; so $Z$ must have another component. Suppose $z \in Z - \{E_n \cup F_x\}$; then $C = \overline{Bz}$ is contained in $Z$. Clearly $C$ is a section of $\pi_n: \mathbf{F}_n \to \mathbf{P}^1$, and by hypothesis it in not a section of $\mathcal{O}(n)$; thus it is a section of $\pi_n$ which intersects $E_n$ at the point $x$. We have $Z = E_n \cup F_x \cup C$, since $\mathbf{F}_n - \{E_n \cup F_x \cup C\} \cong k \times k^*$.

LEMMA 2.6.

(i) *Suppose* $c = 2(n+1)$. *Then there is exactly one embedding of* $B/\Gamma$ *into* $\mathbf{F}_n$ *of Case 3 with* $C \cdot E_n = 1$. *Also for this embedding there is a unique fixed point.*

(ii) *If* $c \neq 2(n+1)$ *there is no such embedding with* $C \cdot E_n = 1$.

*Proof.* Recall from section 1 that one obtains $\mathbf{F}_{n+1}$ from $\mathbf{F}_n$ by blowing up a point $x$ on $E_n$ and contracting the strict transform of the fibre containing $x$.

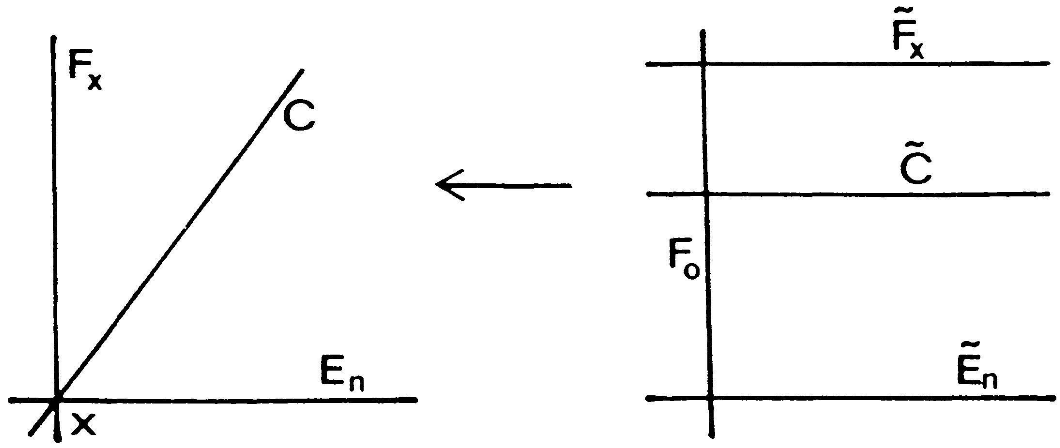


FIGURE 4.

Now suppose we have such an embedding with $C \cdot E_n = 1$. We blow up the point $x$. (See Fig. 4.) Now there are three fixed points on the exceptional divisor $F_0$, so $B$ acts trivially on $F_0$. Blow down $\tilde{F}_x$; We obtain an embedding into $\mathbf{F}_{n+1}$ as in Case 2, where $B$ acts trivially on the fixed fibre. As we have seen in the remark of Case 2, this happens in exactly one case with $c = 2(n+1)$. Conversely, given this embedding into $\mathbf{F}_{n+1}$, by doing the reverse procedure, one obtains exactly one embedding of this type. (By changing the fixed point which is blown up first one obtains an equivalent embedding.) This proves everything except the unicity of the fixed point.

Now we exhibit explicitly the embedding of (i). We use the notation of Lemmas 2.3 and 2.5. Consider the $B$-module $S^{n+1}(k^2)$. We have $B \to PGL(n+2)$ by

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \mapsto \left[ \rho_{n+1} \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \right]$$

where $\rho_{n+1}$ is the $(n+2)$-dimensional irreducible representation of $SL(2, k)$. Consider the closure of the orbit of $x^{n+1} + y^{n+1}$ by $B$ using the basis $\{\binom{n+1}{i} x^i y^{n+1-i}\}_{i=0,\dots,n+1}$. This is exactly

$$X_n = \{(z_0 : s^n : s^{n-1}t : \dots : t^n) \mid z_0, s, t \in k\}.$$

The vertex $(1:0:\dots:0)$ is fixed by this action. The two stable curves in $X_n$ are the "fibre" $\{(z_0 : z_1 : 0 : \dots : 0)\}$ and $\{(s^{n+1} : s^n t : \dots : t^{n+1})\}$, the image of the $(n+1)$-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^{n+1}$. It is easy to see that the isotropy group of $(1:0:\dots:0:1)$ is the finite subgroup of $T$ of order $c$; so this action gives a $B/\Gamma$-embedding into $X_n$ which induces an embedding into $\mathbf{F}_n$. Since the only fixed point on $X_n$ is the vertex and there is only one fixed "fibre", we have exactly one fixed point for the action on $\mathbf{F}_n$. It is easily checked that the intersection number of $E_n$ with the other stable section in $\mathbf{F}_n$ is 1. Thus the lemma is proven. □

LEMMA 2.7. *Any embedding of Case 3 must have* $C \cdot E_n = 1$.

*Proof.* The intersection number $C \cdot E_n = p$ is strictly positive. Suppose that $p > 1$. Now blow up $x$ and then contract the strict transform of $F_x$; we obtain an embedding into $\mathbf{F}_{n+1}$. Let $C_1$ be the strict transform of $C$ in $\mathbf{F}_{n+1}$; then the intersection number $C_1 \cdot E_{n+1}$ is $p - 1$. Also, this new embedding has at least two fixed points: one on $E_{n+1}$ and the other the image of the strict transform of $F_x$ in $\mathbf{F}_{n+1}$. By doing this process $p - 1$ times, we get an embedding into $\mathbf{F}_{n+p-1}$ of Case 3 with $C_{p-1} \cdot E_{n+p-1} = 1$ and at least two fixed points. By Lemma 2.6 this is impossible. Therefore we must have $p = 1$. (See Fig. 5.) □

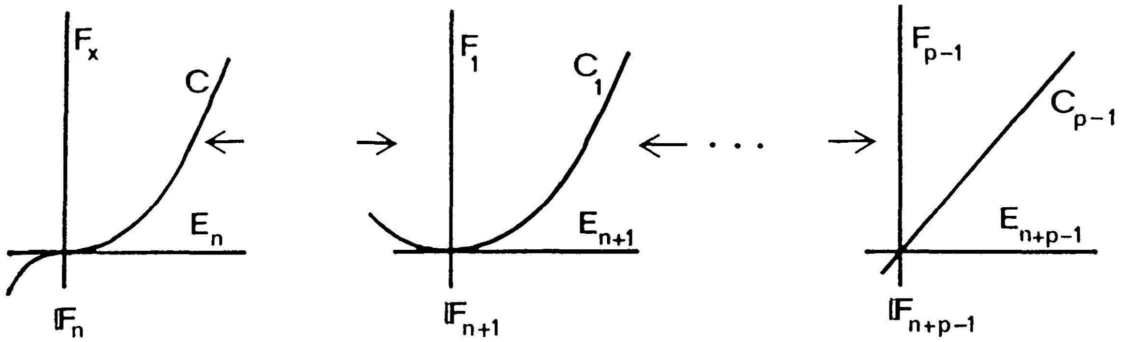


FIGURE 5.

This finishes Case 3. Thus we know all the embeddings into $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{F}_n$, $n \geqslant 1$. The comments after Theorem 2.1 are easily verified by checking each embedding. This finishes the proof of the theorem. □

*Remarks.*

(1) Note that — as to be expected — all the embedding into $\mathbf{F}_1$ are obtained by blowing up the embeddings into $\mathbf{P}^2$ at fixed points.

(2) The "exceptional" embeddings, i.e. those with only one fixed point, are of special interest, because this phenomenon does not occur for smooth complete embeddings of tori. (See [KKMS] for a reference on torus embeddings.)

## § 3. Application to $SL(2)$-embeddings

In [LV] a combinatorical method is presented in order to classify all normal $SL(2)$-embeddings. A natural question is how to classify those which are smooth and complete to obtain a *geometrical* realization. We now sketch how the result of this article is useful for this. (For further details see [JM].)

Given a $B/\Gamma$-embedding $X$, we construct an $SL(2)/\Gamma$-embedding in the following way. Consider the $B$-action on $SL(2) \times X$ given by

$$b \cdot (s, x) = (sb^{-1}, bx)$$

where $b \in B$, $s \in SL(2)$, and $x \in X$. Denote by $SL(2) *_B X$ the variety obtained by quotienting by this action. The action of $SL(2)$ on this variety by left multiplication endows it with the structure of an $SL(2)/\Gamma$-embedding. The projection $SL(2) \times X \to SL(2)$ induces a locally trivial fibre bundle $SL(2) *_B X \xrightarrow{p} SL(2)/B \cong \mathbf{P}^1$. The morphism $p$ is $SL(2)$-equivariant, and the fibre of $p$ is $B$-isomorphic to $X$. So we see that for studying the geometry of the $SL(2)/\Gamma$-embeddings of this form it is useful to study the $B/\Gamma$-embeddings.

As for general $SL(2)/\Gamma$-embeddings one finds the following essential result. Let $\Gamma$ be a finite cyclic subgroup of $SL(2)$. Let $V$ be a smooth $SL(2)/\Gamma$-embedding with orbit $Y$. Then there exists a Borel subgroup $B$ of $SL(2)$ containing $\Gamma$ and an $SL(2)$-stable open neighborhood of $Y$ in $V$ which is of the form $SL(2) *_B X$ for some smooth $B/\Gamma$-embedding $X$. Thus all smooth $SL(2)/\Gamma$-embeddings are *locally* of the form above. Also any smooth $B/\Gamma$-embedding can be completed to a smooth embedding. Thus it is enough to study the complete ones.