

4. Application to Systems of Algebraic Equations

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3.8. *Remark.* Let G be a Gröbner basis of an ideal J . We shall say that G is "simplified" if all $P \in G$ fulfill the following two conditions:

$$\text{lc}(P) \text{ generates the ideal } {}_R \langle \text{lc}(Q) \mid Q \in J, \deg(Q) = \deg(P) \rangle$$

and

$$\text{in}(P) \notin \langle \text{in}(G - \{P\}) \rangle.$$

It is easy to see that the elements of a simplified Gröbner basis have pairwise different degrees.

If R is a field then G is simplified iff the elements of G have pairwise different degrees and $\deg(G)$ is the set of minimal elements (with respect to the natural partial ordering on $\mathbb{N}^n$) in $\deg(J)$.

If G is not simplified, then in the following way we can construct (in a finite number of steps) a simplified Gröbner basis of J :

For every $P \in G$ choose an admissible combination P' of G such that $\deg(P) = \deg(P')$ and $\text{lc}(P')$ generates the ideal

$${}_R \langle \text{lc}(Q) \mid Q \in J, \deg(Q) = \deg(P) \rangle.$$

Then $G' := \{P' \mid P \in G\}$ is a Gröbner basis of J , since $\langle \text{in}(J) \rangle = \langle \text{in}(G) \rangle \subseteq \langle \text{in}(G') \rangle \subseteq \langle \text{in}(J) \rangle$.

If there is a $P' \in G'$ with $\text{in}(P') \in \langle \text{in}(G' - \{P'\}) \rangle$, then $G' - \{P'\}$ is a Gröbner basis, since then $\langle \text{in}(G' - \{P'\}) \rangle = \langle \text{in}(G') \rangle = \langle \text{in}(J) \rangle$.

Replace G' by $G' - \{P'\}$. After finitely many eliminations of this kind we obtain a simplified Gröbner basis.

In example 3.7. the Gröbner basis F_2 is not simplified, since $\text{in}(P_2) = -X_2 \text{in}(P_3)$ and $\text{in}(P_4) = 2X_2 \text{in}(P_5)$. $\{P_1, P_3, P_5\}$ is a simplified Gröbner basis of the ideal generated by F_2 .

4. APPLICATION TO SYSTEMS OF ALGEBRAIC EQUATIONS

Let J be an ideal in $R[X]$, generated by a subset $F \neq \{0\}$.

4.1. We may consider F as a system of algebraic equations in n variables. We denote by K an algebraic closure of the quotient field of R .

Let $Z(F)$ (resp. $Z_K(F)$) be the set $\{z \in R^n$ (resp. K^n) $\mid P(z) = 0$ for all $P \in F\}$ of common zeros in R^n (resp. K^n) of the elements of F . Clearly $Z(F) = Z(J)$ and $Z_K(F) = Z_K(J)$.

4.2. PROPOSITION. Let G be a Gröbner basis of J .

1) $Z_K(J) = \emptyset$ iff $G \cap R \neq \emptyset$.

2) The set $Z_K(J)$ is finite iff $\mathbb{N}^n - \mathcal{D}(G)$ is finite. In this case the cardinality of $Z_K(J)$ is smaller than or equal to the cardinality of $\mathbb{N}^n - \mathcal{D}(G)$.

Proof.

1) By Hilbert's Nullstellensatz we know:

$Z_K(J) = \emptyset$ iff $J \cap R \neq \emptyset$. Therefore $Z_K(J) = \emptyset$ implies $0 \in \deg(J)$, hence $G \cap R \neq \emptyset$.

2) Let I be the ideal generated by J in $K[X]$. Then F is a Gröbner basis of I , too. Again by Hilbert's Nullstellensatz the dimension (as K -vector space) of $K[X]/I$ is an upper bound for the cardinality of $Z_K(J) = Z_K(I)$, and this dimension is finite iff $Z_K(J)$ is so. Since G is a Gröbner basis of I , one easily verifies that the residue classes $X^\alpha + I$, $\alpha \in \mathbb{N}^n - \mathcal{D}(G)$, form a K -basis of $K[X]/I$. This proves the proposition.

4.3. PROPOSITION. Let G be a Gröbner basis of J with respect to the lexicographic ordering (see 1.2.).

If $J \cap R[X_k, \dots, X_n] \neq \{0\}$, then

$$G_k := G \cap R[X_k, \dots, X_n]$$

is a Gröbner basis of

$$J_k := J \cap R[X_k, \dots, X_n];$$

in particular, G_k generates the ideal $J_k \leq R[X_k, \dots, X_n]$ ($1 \leq k \leq n$).

Proof. Let $Q \in J_k$. For any $P \in R[X]$ with $\deg(P) \leq \deg(Q)$ we have $P \in R[X_k, \dots, X_n]$, since $<$ is the lexicographic ordering. By 2.2. and 2.5. there are $c(\alpha, P) \in R$ such that $Q = \sum_{P \in G, \alpha \in \mathbb{N}^n} c(\alpha, P) X^\alpha P$ and $c(\alpha, P) \neq 0$ implies $\deg(X^\alpha P) \leq \deg(Q)$.

Hence we have $X^\alpha P \in R[X_k, \dots, X_n]$ for $c(\alpha, P) \neq 0$, and, by 2.5. again, G_k is a Gröbner basis of J_k .

4.4. Now we can apply the theory of Gröbner bases to find the solutions to the system F of algebraic equations. Consider the following algorithm:

First we construct a Gröbner basis G of J with respect to the lexicographic ordering (see 3.6.). As in 4.3. we write G_k for $G \cap R[X_k, \dots, X_n]$, $1 \leq k \leq n$.

Compute the greatest common divisor P_n of the (univariate) polynomials in G_n . Find a zero $a_n \in R$ of P_n . If P_n has no zero in R , then $Z(J) = \emptyset$.

Let $k \in \{1, \dots, n-1\}$. Suppose that $a_{k+1}, \dots, a_n \in R$ have already been found. Let $G_k(a_{k+1}, \dots, a_n) \subseteq R[X_k]$ be the set of polynomials in one variable X_k obtained from G_k by substituting everywhere a_j for X_j , $k+1 \leq j \leq n$.

Compute the greatest common divisor P_k of the polynomials in $G_k(a_{k+1}, \dots, a_n)$. Find a zero $a_k \in R$ of P_k . If P_k has no zero in R , we have to go back to G_n and to find another sequence $a'_n, \dots, a'_{k+1}$.

If we obtain $(a_1, \dots, a_n)$ by this algorithm, it is an element of $Z(J)$. By 4.3. all elements of $Z(J)$ can be computed in this way.

Suppose that $Z_K(J)$ is finite (i.e. $\mathbf{N}^n - \mathcal{D}(G)$ is finite) and that we are able to solve univariate polynomial equations in R (which is the case for $R = \mathbf{Z}$). Then the algorithm above yields $Z(J)$ in a finite number of steps.

4.5. *Example.* Let F be the subset

$$\begin{aligned} &\{2X_1^4 + 3X_1^3X_2X_3 - X_1X_2^2 + 5X_1 - 3X_2^2 - 5X_2X_3 - 2X_3 + 41, \\ &4X_1^4 + 6X_1^3X_2X_3 - 2X_1X_2^2 + 10X_1 + 3X_2^2 + 5X_2X_3 + 2X_3^3 - 11X_3^2 + 19X_3 + 25, \\ &6X_2^2 + 10X_2X_3 + 2X_3^3 - 11X_3^2 + 21X_3 - 40\} \quad \text{of} \quad \mathbf{Z}[X_1, X_2, X_3]. \end{aligned}$$

By the algorithm 3.6. we get a Gröbner basis G of the ideal generated by F :

$$\begin{aligned} G = &\{2X_3^3 - 11X_3^2 + 17X_3 - 6, \\ &3X_2^2 + 5X_2X_3 + 2X_3 - 17, \\ &2X_1^4 + 3X_1^3X_2X_3 - X_1X_2^2 + 5X_1 + 24\}. \end{aligned}$$

Now $Z(G_3) = \{2, 3\}$, $Z(G_2(2)) = \{1\}$, $Z(G_2(3)) = \emptyset$ and $Z(G_1(1, 2)) = \{-2\}$. So $Z(F) = \{(-2, 1, 2)\}$.

5. APPLICATION TO A GEOMETRIC PROBLEM

5.1. For $P \in R[X]$ let $\tilde{P}$ be the homogeneization of P by a further variable X_{n+1} . For an ideal $J \subseteq R[X]$ we write $\tilde{J}$ for the ideal generated by $\{\tilde{P} \mid P \in J\}$ in $R[X_1, \dots, X_{n+1}]$.

PROPOSITION. Let G be a Gröbner basis of J with respect to the graded inverse lexicographic ordering (see 2.1.). Then $\tilde{G} := \{\tilde{P} \mid P \in G\}$ is a Gröbner basis of $\tilde{J}$.