

## §2. Topological Buildings

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$y = s_1 \cdots s_k (s_i \in S)$  and  $x$  has a reduced decomposition obtained by deleting some subset of the  $s_i$ 's occurring in  $y$ . (For a very nice account of these related matters, see [14]). If  $W$  is finite,  $W$  has a unique element  $w_0$  of maximal length, we define the length of  $W$  to be  $l(w_0)$ .

## § 2. TOPOLOGICAL BUILDINGS

A *Tits system*  $(G, B, N, S)$  consists of a group  $G$ , subgroups  $B$  and  $N$ , and a set  $S$ , which satisfy the following axioms:

- (2.1)  $B \cap N$  is normal in  $N$ , and  $S$  is a set of involutions generating  $\bar{W} \equiv N/B \cap N$ ,
- (2.2)  $B$  and  $N$  generate  $G$ ,
- (2.3) If  $s \in S$ ,  $sBs \neq B$ ,
- (2.4) if  $s \in S$ ,  $w \in W$ , then  $sBw \leq BwB \cup Bs w B$ .

(The use of expressions such as  $sBw$  is a standard abuse of notation).

*Example.* Let  $G$  be a reductive algebraic group over an algebraically closed field (e.g.,  $GL(n, \mathbf{C})$ ), let  $B$  be a Borel subgroup (e.g. upper triangular matrices), and let  $N$  be the normalizer of a maximal torus (that lies in  $B$ ). This data determines a set  $S$  of simple reflections generating the Weyl group  $W$  (e.g., the usual generators  $s_1, \dots, s_{n-1}$  of  $\Sigma_n$ ). Then one of the main results in the structure theory of reductive groups is that  $(G, B, N, S)$  is a Tits system (see for example [15]).

Throughout this paper we will assume that the set  $S$  is finite; its cardinality  $l$  is the *rank* of the system.

We next list some of the important properties of a Tits system.

- (2.5) (Bruhat Decomposition)  $G = \coprod_{w \in W} BwB$  (disjoint union),
- (2.6)  $(W, S)$  is a Coxeter system.

A subgroup  $P$  of  $G$  is *parabolic* if it contains a conjugate of  $B$ . In particular if  $I \subseteq S$ , the subgroup  $P_I$  generated by  $B$  and  $I$  is parabolic.

- (2.7) (a) The parabolic subgroups containing  $B$  are precisely the  $P_I$ ,  $I \subseteq S$ . No two of these are conjugate; in particular there are exactly  $2^l$  such subgroups, which form a lattice isomorphic to the lattice of subsets of  $S$ .
  - (b)  $P_I = BW_I B$
  - (c) Every parabolic  $P$  is self-normalizing:  $N_G P = P$ .

(2.8) (Bruhat decomposition, general version)  $G = \coprod_{w \in W_I \backslash W/W_J} P_I w P_J$  (disjoint union).

The next result, which we will refer to as the *Steinberg Lemma*, is somewhat technical; however it is not hard to prove and is extremely useful. It is a mild generalization of Theorem 15 of [32] and Proposition 3.1 of [19].

(2.9) Let  $I \subseteq S$  and suppose  $w$  is the unique element of minimal length of  $wW_I$ . Suppose  $w = w_1 \dots w_k$  where  $l(w) = l(w_1) + \dots + l(w_k)$ . Then

(a) If  $Y_i$  is any subset of  $Bw_iB$  such that  $Y_i \rightarrow Bw_iB/B$  is bijective (resp. surjective) ( $1 \leq i \leq k$ ), then  $Y_1 \times Y_2 \dots \times Y_k \rightarrow BwP_I/P_I$  is bijective (resp. surjective).

(b) Suppose  $w_i \in S$ ,  $1 \leq i \leq k$  i.e.,  $w_1 \dots w_k$  is a reduced decomposition of  $w$ . Let  $Z_i$ ,  $1 \leq i \leq k$ , be any subset containing 1 of  $P_{w_i}$  such that  $Z_i \rightarrow P_{w_i}/B$  is surjective. Then the image of  $Z_1 \times \dots \times Z_k \rightarrow G/P_I$  is  $\coprod_{x \leq w} BxP_I/P_I$ .

The maps in (a), (b) are the obvious multiplication/projection maps. Part b refers to the Bruhat order on  $W^I$ .

(2.10) *Remark.* The Tits system of a reductive algebraic group has several additional features:  $B = HU$ , where  $H$  is a maximal torus and  $U$  is a normal unipotent subgroup,  $U$  in turn is described in terms of its root subgroups, and there is an "opposite" Borel subgroup  $B^-$  such that  $B \cap B^- = H$ . This additional structure can also be axiomatized in an elegant way, leading to the "refined" Tits system of Kac and Peterson [19]. One then obtains, for example, the *Birkhoff decomposition*  $G = \coprod_{w \in W} B^- w B$  as a consequence of the axioms.

We now define a *topological Tits system* to be a Tits system such that  $G$  is a topological group,  $B$  and  $N$  are closed subgroups, and  $W$  is discrete (i.e.  $N \cap B$  is an open subgroup of  $N$ ). We will usually also assume (for reasons which will be apparent shortly):

(2.11) *Axiom.* If  $I$  is a proper subset of  $S$ ,  $W_I$  is finite.

This axiom is satisfied if  $W$  is an irreducible affine Weyl group, or finite. To get any interesting results some further axiom seems necessary. One direction is considered in [11], where the groups in question are algebraic groups over local fields, with the valuation topology. Here, with loop groups in mind, the following axiom seems efficient:

(2.12) *Axiom.* For each  $s \in S$  there is a subset  $A_s$  of  $P_s$  such that (a)  $A_s B = P_s$ , (b)  $A_s$  is compact and contains 1, and (c)  $A_s = \overline{A_s \cap BsB}$ . This axiom is motivated by Steinberg's approach [32].

(2.13) PROPOSITION. Let  $(G, B, N, S)$  be a topological Tits system satisfying (2.12). Then

(a)  $\overline{BwB} = \coprod_{x \leq w} BxB (w \in W)$ . More generally if  $I \leq S$ , and  $w \in W^I$ ,  $\overline{BwP_I} = \coprod_{x \leq w} BxP_I$  (here  $x \in W^I$ ),

(b)  $B$ -orbits in  $G/P_I$  are locally closed,

(c) If  $W$  satisfies (2.11), parabolic subgroups are closed.

*Proof.* First we show  $P_s = \overline{BsB}$ . Since  $P_s = A_sB$ , with  $A_s$  compact and  $B$  closed,  $P_s$  is closed, so  $P_s \supseteq \overline{BsB}$ . But also  $B \subset P_s = A_sB \subset \overline{BsB}$ , which proves our claim. Part (a) now follows easily from the Steinberg lemma: Let  $M_w = \coprod_{x \leq w} BxP_I$ , and let  $w = s_1 \cdots s_k$  be a reduced decomposition. Then  $M_w = A_1 \cdots A_k P_I$  and hence is closed. Next, suppose  $x \leq w$ ; we must show  $BxB \leq \overline{BwB}$ . It is enough to consider the case when  $X$  has a reduced decomposition  $x = s_1 \cdots \hat{s}_i \cdots s_k$  (omit  $s_i$ ). Then

$$BxP_I = A'_1 \cdots A'_{i-1} A'_{i+1} \cdots A'_k P_I \leq A'_1 \cdots A'_{i-1} \bar{A}_i \cdots A'_k P_I \leq \overline{BwP_I}$$

(since  $1 \in A_i$ ), where  $A'_i = A_i \cap Bs_iB$ . This proves (a). Part (b) is immediate since the complement of  $BwP_I$  in its closure is a finite union of sets of the form  $M_x$ , hence is closed. Since  $P_I = BW_I B$ , (c) is also immediate from (a) if  $W_I$  is finite.  $\square$

From now on we will assume 2.11 and 2.12. The homogeneous spaces  $G/P_I$  will be called *flag spaces*. The  $B$ -orbits  $E_w = BwP_I/P_I$  are *Schubert strata* and the compact subspaces  $\overline{E_w}$  are *Schubert subspaces*.

We next consider the *building*  $\mathcal{B}_G$  associated to a topological Tits system  $(G, B, N, S)$ . (The notation is ambiguous—indeed in the case of loop groups,  $G$  will support two natural but totally different Tits system. However the system we have in mind will be clear from the context.) In the discrete case,  $\mathcal{B}_G$  is usually defined as the following simplicial complex. The vertices are the maximal (proper) parabolics, and  $P_1 \cdots P_k$  span a simplex if  $\bigcap_{i=1}^k P_i$  contains a conjugate of  $B$ . In general it is convenient to reinterpret this definition as follows: first of all, by definition every parabolic  $P$  is conjugate to a unique  $P_I$ ; we say that  $P$  has type  $I$ . Thus the maximal parabolics are the parabolics of type  $[s]$ , where  $[s] = S - \{s\}$ . More generally the  $k$ -simplices correspond to the parabolics of type  $I$ , where  $|I| = l - k - 1$ . Thus the simplices all have dimension  $\leq l - 1$ , with the  $l - 1$  simplices corresponding to the conjugates of  $B$ . Furthermore, in view

of 2.7 (c), the set of parabolics of type  $I$  is canonically identified with  $G/P_I - xP_I$  corresponding to  $xP_Ix^{-1}$ . One can easily check that with this interpretation, a simplex  $xP_I$  is a face of a simplex  $yP_J$  if and only if  $I \supset J$  and  $xP_I = yP_I$ . In particular, every simplex is a face of some  $l - 1$  simplex. Hence, as a set,  $B_G$  can be identified with  $G/B \times \Delta/\sim$ , where  $\Delta$  is the  $l - 1$  simplex with vertex set  $S$ , and  $(g_1B, X_1) \sim (g_2B, X_2)$  if  $X_1 = X = X_2$ ,  $X \in \Delta_I$ , and  $g_1P_I = g_2P_I$ . (Here  $\Delta_I$  is the face of  $\Delta$  corresponding to  $I \leq S$ .) We will therefore *define* the building  $\mathcal{B}_G$  associated to the topological Tits system  $(G, B, N, S)$  to be  $G/B \times \Delta$  modulo this equivalence relation, with the quotient topology.

*Remark.* Another way of expressing this is as follows: Let  $C$  be the category defined by the poset of proper subsets of  $S$  (including the empty set). We have a functor from  $C$  to topological spaces given by  $I \mapsto G/P_I$ . Then  $\mathcal{B}_G$  is precisely the homotopy colimit of this diagram of spaces, in the sense of [8], p. 327 ff.

(2.14) PROPOSITION. *The equivalence relation on  $G/B \times \Delta^{l-1}$  is generated by the relations  $(g_1B, X) \sim (g_2B, X)$  if  $X$  lies on the wall  $\Delta_s$  and  $g_1P_s = g_2P_s$ .*

*Proof.* In the usual language, (2.14) is the assertion that any two chambers are linked by a "gallery". (See e.g. [11], appendix.) Since the action of  $G$  on  $G/B$  induces a well defined action on  $\mathcal{B}_G$ , we are reduced to showing that if  $(B, X) \sim (gB, X)$ —i.e.  $X \in \Delta_I$  and  $g \in P_I$ —then  $(B, X)$  and  $(gB, X)$  are linked by a sequence of relations of the stated type. But  $gB = bwB$  with  $w \in W_I$ ; hence if  $w = s_1 \cdots s_k$  is a reduced decomposition, the elements  $(B, X), (bs_1B, X), (bs_1s_2B, X), \dots, (bwB, X)$  provide the desired sequence.  $\square$

Note that the set  $\Delta$  is a fundamental domain for the action of  $G$  on  $\mathcal{B}_G$ . On the other hand, it is easy to check that the closed subspace  $\mathcal{B}_W$  consisting of the pairs  $(wB, X)$ ,  $w \in W$ , is a fundamental domain for the  $B$  action. (The point is that if  $bw_1P_I = w_2P_I$ , then  $w_1P_I = w_2P_I$ , by the Bruhat decomposition.) This space  $\mathcal{B}_W$ , which we will call the *foundation* of the building, is a simplicial complex since  $W$  is discrete. Since it will turn out that  $\mathcal{B}_G$  is in a sense a "thickening" of the foundation, the following well known description of  $\mathcal{B}_W$  may be of interest.

(2.15) PROPOSITION. Suppose  $\Phi$  is an irreducible root system in the Euclidean space  $V$ . Then

(a) If  $W$  is the affine Weyl group associated to  $\Phi$ , then  $\mathcal{B}_W$  is isomorphic as a simplicial  $W$ -complex to  $V$  (triangulated by the hyperplanes of  $\Phi$ ).

(b) If  $W$  is the Weyl group of  $\Phi$ ,  $\mathcal{B}_W$  is isomorphic as simplicial  $W$ -complex to the unit sphere of  $V$ , triangulated by the Weyl chambers. More precisely,  $\mathcal{B}_W$  can be identified with the  $W$  orbit of the outer wall of the Cartan simplex.

*Proof.* For (a), map  $W \times \Delta \xrightarrow{\varphi} V$  by identifying  $\Delta$  with the Cartan simplex in  $V$  and using the action map. Then  $\varphi$  is onto (1.1) and furthermore  $\varphi(w_1, x) = \varphi(w_2, X_2)$  if and only if  $X_1 = X = X_2$ ,  $X \in \Delta_I$ , and  $w_1 = w_2$  modulo the isotropy group of  $X$ . But this isotropy group is precisely  $W_I$  (1.2), so  $\varphi$  factors through the desired isomorphism  $\mathcal{B}_W \rightarrow V$ . The proof of (b) is similar.  $\square$

We now come to the main result of this section. Filter  $G/B$  by  $F_k(G/B) = \coprod_{l(w) \leq k} E_w$ . Similarly,  $\mathcal{B}_G$  is filtered by  $F_k(\mathcal{B}_G) = F_k(G/B) \times \Delta / \sim$ .

(2.16) THEOREM. Let  $(G, B, N, S)$  be a topological Tits system which either is discrete or satisfies (2.11) and (2.12). Assume also that the inclusions  $F_k(B_G) \subset F_{k+1}(B_G)$  are cofibrations. Then

(a) If  $W$  is infinite,  $\mathcal{B}_G$  is contractible.

(b) If  $W$  is finite of length  $r$ ,  $\mathcal{B}_G$  is homotopy equivalent to the  $(l-1)$ st suspension  $S^{l-1} \wedge (F_r(G/B)/F_{r-1}(G/B))$ .

*Remark.* If  $G$  is discrete,  $F_k \mathcal{B}_G$  is a subcomplex of the simplicial complex  $\mathcal{B}_G$ , so the cofibration hypothesis is automatically satisfied. Furthermore if  $W$  is finite the smash product in (b) is just a wedge of  $|F_r G/B - F_{r-1} G/B|$   $(l-1)$ -spheres. This case is due to Solomon and Tits; cf. [11].

*Proof of (2.16).* Let  $X_k$  denote  $F_k \mathcal{B}_G / F_{k-1} \mathcal{B}_G$ , and let  $X'_k = F_k(G/B) / F_{k-1}(G/B)$ . Then we will show

(2.17) If  $k$  is less than the length of  $W$ ,  $X_k$  is contractible. If  $k = r = \text{length of } W$ ,  $X_k$  is homeomorphic to  $(F_r(G/B)/F_{r-1}(G/B)) \wedge S^{l-1}$ .

If  $W$  is infinite, it follows that  $F_k \mathcal{B}_G$  is contractible for all  $k$ , and hence  $\mathcal{B}_G$  is contractible. If  $W$  is finite, part (b) of the theorem is also immediate.

To prove 2.17, first consider the quotient map  $\pi: F_k(G/B) \times \Delta \rightarrow X_k$ . In fact  $\pi$  is merely collapsing a subspace to a point:

(2.18) Let  $A_1 = (b_1w_1B, X_1)$ ,  $A_2 = (b_2w_2B, X_2)$ . If  $\pi(A_1) = \pi(A_2)$ , then either  $A_1 = A_2$  or  $\pi(A_1) = \pi(A_2) = *$  ( $*$  is the basepoint  $F_{k-1}B_G$ ).

For suppose  $\pi(A_1) \neq *$ , and  $X_1 = \Delta_I$ . Then  $l(w_1) = k$  and  $w_1 \in W^I$ . This forces  $X_1 = X_2$  and  $w_1 = w_2 \text{ mod } W_I$ ; hence  $w_1 = w_2$  since  $l(w_2) \leq k$  by assumption. Then  $b_1w_1P_I = b_2w_1P_I$ . But whenever  $w \in W^I$ ,  $b_1wP_I = b_2wP_I$  implies  $b_1wB = b_2wB$  (easy exercise).

It now follows that  $X_k = \bigvee_{l(w)=k} X_w$ , where  $X_w$  is the image of  $\bar{E}_w \times \Delta$  in  $X_k$ , and to prove (2.17) we need only consider a fixed  $X_w$ . Let  $X'_w = \bar{E}_w / (\bar{E}_w - E_w)$ , and let  $\Delta'$  be the subcomplex of  $\Delta$  consisting of the walls  $\Delta_s$  such that  $l(ws) < l(w)$ . Then (2.18) implies:

$$(2.19) \quad X_w = X'_w \wedge (\Delta/\Delta').$$

For  $X_w$  is  $\bar{E}_w \times \Delta$  modulo the subspace of points which are equivalent (in  $\mathcal{B}_G$ ) to a point of lower filtration, namely,  $\bar{E}_w \times \Delta' \cup \bar{E}_w - E_w \times \Delta$ . It remains to identify  $\Delta'$ . Since  $F_0\mathcal{B}_G = \Delta$  is contractible, we may assume  $k \geq 1$ ; then  $\Delta'$  is nonempty. If  $k < l(W)$ , then there is at least one  $s \in S$  such that  $l(ws) > l(w)$ ; hence  $\Delta'$  is not the entire boundary of  $\Delta$  and  $\Delta/\Delta'$  is contractible. If  $k = l(W)$ , then  $w$  is unique,  $\Delta' =$  boundary of  $\Delta$ , and  $\Delta/\Delta' = S^{l-1}$ . This completes the proof of (2.17), and of the theorem. □

*Remark.* Our proof of Theorem 2.16 is an adaptation of the standard (discrete) proof to the topological setting. Much of the proof depends only on the Weyl group  $W$ , and indeed shows e.g. for  $W$  infinite that the foundation of the building is contractible. In fact the deformation of  $F_k(\mathcal{B}_W)$  into  $F_{k-1}(\mathcal{B}_W)$  has the property that the isotropy group in  $B$  of a point  $X$  in  $\mathcal{B}_W$  is an increasing function of time, and hence extends uniquely to a  $B$ -equivariant deformation of  $F_k(B_G)$ . In the discrete case this extension is automatically continuous, and shows that Theorem (2.16) holds  $B$ -equivariantly. (This was observed, (not for the first time) in [21], and has an interesting application concerning the Steinberg representation of a finite Chevalley group.) However this proof does not work in the topological case; simple counterexamples show that the extension will be discontinuous.

In many cases the Bruhat decomposition of  $G/P$  is in fact a  $CW$  decomposition. The following axioms are convenient in this regard:

(2.20) *Axiom.* For each  $s \in S$ , the projection  $P_s \rightarrow P_s/B$  has a local section.

(2.21) *Axiom.* For each  $s \in S$ ,  $P_s/B$  is homeomorphic to a sphere of positive dimension.

We then have:

(2.22) THEOREM. Let  $(G, B, N, S)$  be a topological Tits system satisfying axioms 2.11, 2.20 and 2.21. Let  $P \equiv P_I$  be a parabolic subgroup,  $I \leq S$ , and give  $G/P$  the compactly generated topology. Then

(a) Axiom 2.12 is satisfied.

(b) The Bruhat decomposition of  $G/P$  is a CW decomposition, and the closure relations on the cells are given by the Bruhat order on  $W^I$ .

(c) The building  $\mathcal{B}_G$  satisfies the cofibration condition of Theorem 2.16.

*Proof.* By assumption there are maps  $D^{m(s)} \xrightarrow{\varphi_s} P_s/B$  such that  $\varphi_s^{-1}(B) = \partial D^{m(s)}$  and  $D^{m(s)}/\partial D^{m(s)} \rightarrow P_s/B$  is a homeomorphism. Furthermore  $\varphi_s$  lifts to a map  $\tilde{\varphi}_s: D^{m(s)} \rightarrow P_s$  with  $1 \in \tilde{\varphi}_s(\partial D^{m(s)})$ . Thus, in Axiom (2.12) we may take  $A_s = \tilde{\varphi}_s(\mathring{D}^{m(s)})$ , proving (a). Since  $P$  is closed (2.13c),  $G/P$  is a Hausdorff space. If  $w \in W^I$  has reduced decomposition  $w = s_1 \cdots s_k$ , the Steinberg lemma (2.9) shows that the multiplication map  $D^{m(s_1)} \times \cdots \times D^{m(s_k)} \rightarrow \bar{E}_w$  (using  $\tilde{\varphi}_{s_i}$ ) is a characteristic map for the cell  $E_w$ . The boundary of each cell is a finite union of cells of lower dimension by 2.13a, and  $G/P$  has the weak topology by assumption. The closure relations also follow from (2.13). This proves (b). For (c) we observe that  $\mathcal{B}_G$  (with the compactly generated topology) is itself a CW-complex, and the filtrations  $F_k \mathcal{B}_G$  are subcomplexes: Indeed if we regard  $\mathcal{B}_G$  as a quotient space of  $\coprod_{I \leq S} (G/P_I \times \Delta_I)$ , it is clear that there is one cell for each  $I < S$  and  $w \in W^I$ .  $\square$

If  $G, P_I$  are as in the above theorem, and  $w \in W^I$  has reduced decomposition  $w = s_1 \cdots s_k$ , let  $n(w) = n(s_1) + \cdots + n(s_k)$ . Thus  $n(w) = \dim E_w$  and so in particular is independent of the choice of reduced decomposition. Now whenever a space has a locally finite cell decomposition, we have a cell series  $\sum a_i t^i$ , where  $a_i$  is the number of cells of dimension  $i$ . We then have:

(2.23) COROLLARY.  $G/P_I$  admits a CW—decomposition with cell series  $\sum_{w \in W^I} t^{n(w)}$ .  $\square$

Note also:

(2.24) COROLLARY. If  $W$  is finite with maximal length element  $w_0$ ,  $\mathcal{B}_G$  is a sphere of dimension  $n(w_0) + l - 1$ .  $\square$

We conclude this section with two “classical” examples. Let  $G$  be a semisimple compact Lie group and consider the Tits system  $(G, B, N, S)$ , where  $B$  is a Borel subgroup, etc. First we claim that this is a topological Tits system satisfying all four of our axioms. Since  $W$  is finite, (2.11) is



trivially satisfied. In (2.12) we can take  $A_s$  to be the “little  $SU(2)$ ” (or  $PSU(2)$ )  $G_s$  ( $P_s$  has Iwasawa decomposition  $P_s = G_s B$ ). In any case there is a commutative diagram

$$\begin{array}{ccc} G_s & \rightarrow & P_s \\ \downarrow & & \downarrow \\ CP^1 = G_s/G_s \cap T & = & P_s/B \end{array}$$

which proves (2.20), (2.21), and hence (2.12) simultaneously. The Bruhat decomposition of  $G_{\mathbb{C}}/P_I$ ,  $P_I$  parabolic, is then the classical Schubert cell decomposition of the flag variety  $G_{\mathbb{C}}/P_I$ . We have  $n(s) = 2$  for all  $s$ , so  $n(w) = 2l(w)$  for all  $w \in W^I$ . In particular the associated building  $\mathcal{B}_{G_{\mathbb{C}}}$  is a sphere of dimension  $2l(w_0) + l - 1$  (since  $l(w)_0$  is the number of positive roots, this is exactly  $\dim G - 1$ ).

The second example (which is a generalization of the first) involves symmetric spaces  $G/K$  and the associated semisimple real Lie group  $G_{\mathbb{R}}$  as in § 1. Thus  $G_{\mathbb{R}}$  is the fixed group of the involution  $\sigma$  on  $G_{\mathbb{C}}$ . Now  $\sigma$  need not preserve the Borel subgroup  $B$  of  $G_{\mathbb{C}}$ , but it does preserve the parabolic  $Q$  associated to the black nodes of the Satake diagram. We will write  $B_{\mathbb{R}}, N_{\mathbb{R}}, W_{\mathbb{R}}, S_{\mathbb{R}}$  for  $Q^{\sigma}, N_{K^t m}, W_{G/K}, S_{G/K}$ , respectively.

(2.25) THEOREM.  $(G_{\mathbb{R}}, B_{\mathbb{R}}, N_{\mathbb{R}}, S_{\mathbb{R}})$  is a topological Tits system satisfying the four axioms. □

A proof that this is a Tits system can be found in [33]. The parabolic subgroups of  $G_{\mathbb{R}}$  are related in an obvious way to those of  $G_{\mathbb{C}}$ : Given  $I \subset S_{\mathbb{R}}$ , let  $I'$  be the corresponding set in  $S$  (see § 1). We denote by  $\mathcal{O}_I$  the parabolic in  $G_{\mathbb{R}}$  generated by  $B_{\mathbb{R}}$  and  $I$ . Then  $\mathcal{O}_I = (P_{I'})^{\sigma}$ . ( $B_{\mathbb{R}}$  is usually called a “minimal parabolic”, but this terminology conflicts with our use of the term. From the point of view of Tits systems, it is precisely analogous to the Borel subgroup of  $G_{\mathbb{C}}$ —although in general it is neither solvable nor connected.) The rest of the theorem is also easily deduced from [33]; the details will be omitted, but see § 5. The main point is that for the minimal parabolics  $\mathcal{O}_i$ ,  $\mathcal{O}_i/B_{\mathbb{R}}$  is a sphere of dimension  $n_i$ .

As for the building, one can deduce from (2.24) that it is a sphere whose dimension is  $\dim G/K - 1$ . However it is an interesting fact, that does not seem to appear in the literature, that the building can be canonically identified with the “tangent cut locus” of  $G/K$ : first recall (cf. [10], [20]) that if  $M$  is a compact Riemannian manifold and  $p$  is a fixed point of  $M$ , a point

$x$  is a *cut point* (with respect to  $p$ ) if there is a geodesic from  $p$  to  $x$  that minimizes arc length up to  $x$  but no further. The *cut locus* is the set of cut points. Similarly a vector  $X$  in the tangent space  $T_p$  is a *tangent cut point* if  $\exp_p X$  is a cut point along the geodesic  $\exp_p(tX)$ . The *tangent cut locus* is the set of all such points in  $T_p$ , and is homeomorphic to the unit sphere in  $T_p$ . When  $M = G/K$  we take  $p = 1$ .

(2.26) THEOREM. *Let  $G/K$  be a simply-connected symmetric space, with  $G$  simple. Then the tangent cut locus is precisely the  $K$ -orbit in  $\mathfrak{m}$  of the outer wall of the Cartan simplex  $\Delta_m$ . It is therefore canonically identified with the topological building of the associated real form  $G_{\mathbf{R}}$ .*

As usual, the assumption  $G$  simple is just for convenience. We sketch the proof: the first assertion is a fairly easy consequence of Theorem (1.8), and is left to the reader. Now consider the building  $\mathcal{B}_{G_{\mathbf{R}}}$ . It is a quotient space of  $G_{\mathbf{R}}/B_{\mathbf{R}} \times \Delta_0 = K/C_{Kt_m} \times \Delta_0$ , where  $\Delta_0$  is a simplex of dimension  $(\text{rank } G/K)-1$ ; we take  $\Delta_0$  to be the outer wall of  $\Delta_m$ . For each  $I \leq S_{G/K}$ , let  $\Delta_I$  temporarily denote the corresponding face of  $\Delta_0$ ; i.e.  $\{X \in \Delta_0 : \alpha_i(x) = 0 \forall i \in I\}$ . Then the  $K$ -orbit of  $\Delta_0$  in  $\mathfrak{m}$ ,  $K\Delta_0$ , is also a quotient of  $K/C_{Kt_m} \times \Delta_0$ . The relations are  $(k_1 X) \sim (k_2 X)$  if  $X \in \Delta_I$  and  $k_1 = k_2 \text{ mod } K_I$ . But  $K_I = K \cap \mathcal{O}_I$ , so these relations are identical to the ones that define the building.  $\square$

### § 3. LOOP GROUPS

Let  $LG, LG_{\mathbf{C}}$  denote the free loop spaces. Let  $G_{\mathbf{C}}$  denote the group of loops which are restrictions of regular maps  $\mathbf{C}^* \rightarrow G_{\mathbf{C}}$ , and let  $L_{alg}G = L_{alg}G_{\mathbf{C}} \cap LG$ . Thus if we fix an embedding  $G_{\mathbf{C}} \subset GL(n, \mathbf{C})$ ,  $L_{alg}G$  consists of the loops  $f$  in  $LG$  admitting a finite Laurent expansion  $f(z) = \sum_{k=-m}^m A_k z^k$ , whereas  $L_{alg}G_{\mathbf{C}}$  consists of the loops  $f$  in  $LG_{\mathbf{C}}$  such that both  $f$  and  $f^{-1}$  admit finite Laurent expansions. We will also write  $\tilde{G}_{\mathbf{C}}$  for  $L_{alg}G_{\mathbf{C}}$ . In fact  $\tilde{G}_{\mathbf{C}}$  is the group of points over  $\mathbf{C}[z, z^{-1}]$  of the algebraic group  $G_{\mathbf{C}}$ . Its Lie algebra is the loop algebra  $\tilde{g}_{\mathbf{C}}$  of regular maps  $\mathbf{C}^* \rightarrow g_{\mathbf{C}}$ . The integer  $m$  in the above Laurent expansion defines a filtration of  $\tilde{G}_{\mathbf{C}}$  by finite dimensional subspaces; we give  $\tilde{G}_{\mathbf{C}}$  the corresponding weak topology.

Let  $P$  denote the subgroup of  $\tilde{G}_{\mathbf{C}}$  consisting of regular maps  $\mathbf{C} \rightarrow G_{\mathbf{C}}$  (i.e. maps with nonnegative Laurent expansion, or  $G_{\mathbf{C}[z]}$ ), and let  $\tilde{B}$  denote the Iwahori subgroup:  $\{f \in P : f(0) \in B^-\}$ . Finally, let  $\tilde{N} = L_{alg}N_{\mathbf{C}}$ , and recall that  $\tilde{W}$  can be regarded as a "subgroup" of  $\tilde{G}_{\mathbf{C}}$ , since  $R \leq \text{Hom}(S^1, T) \leq L_{alg}T$ . More precisely, we have  $\tilde{N}/T_{\mathbf{C}} = \hat{W}$ , and  $\tilde{W} \subset \hat{W}$ .