

# 1. Example of a Stein space $X$ with $\widetilde{O(X)} \not\cong O(\widetilde{X})$

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An analytic consequence of the construction presented here is that the normalization  $\tilde{X}$  of an irreducible Stein space  $X$  is  $\widetilde{\mathcal{O}(X)}$ -convex,  $\widetilde{\mathcal{O}(X)}$ -separable and has local coordinates by functions in  $\widetilde{\mathcal{O}(X)}$ . Some algebraic results are that  $\mathcal{O}(\tilde{X})$  is completely normal and that the two algebras  $\widetilde{\mathcal{O}(X)}$  and  $\mathcal{O}(\tilde{X})$  are always locally equal, i.e. their localizations at all maximal ideals in  $\mathcal{O}(X)$  are equal.

In this paper, a complex space refers to a reduced complex space with countable topology.

### 1. EXAMPLE OF A STEIN SPACE $X$ WITH $\widetilde{\mathcal{O}(X)} \neq \mathcal{O}(\tilde{X})$

Let  $(X, \mathcal{O})$  be a complex space with normalization  $\pi: \tilde{X} \rightarrow X$ . Since  $\pi$  is surjective, the map  $\pi^*: \mathcal{O}(X) \rightarrow \mathcal{O}(\tilde{X})$ ,  $f \mapsto f \circ \pi$ , is injective and the holomorphic functions  $\mathcal{O}(X)$  on  $X$  can be considered to be a subring of the holomorphic functions  $\mathcal{O}(\tilde{X})$  on the normalization  $\tilde{X}$  of  $X$ ; this will be indicated by  $\mathcal{O}(X) \subset \mathcal{O}(\tilde{X})$ . If  $X$  is irreducible and Stein, then  $\mathcal{O}(\tilde{X})$  contains the integral closure  $\widetilde{\mathcal{O}(X)}$  of  $\mathcal{O}(X)$  but does not always coincide with it, as will be shown in this section.

For an irreducible complex space  $(X, \mathcal{O})$ , the integral domain  $\mathcal{O}(X)$  is said to be *normal*, if it is integrally closed in its field of fractions  $Q(\mathcal{O}(X))$ , i.e.  $\widetilde{\mathcal{O}(X)} = \mathcal{O}(X)$ . Recall that  $Q(\mathcal{O}(X))$  is the field of meromorphic functions  $M(X)$  on  $X$  when  $X$  is irreducible and Stein due to Theorem B [10, 53.1, 52.17], and that the algebras  $M(X)$  and  $M(\tilde{X})$  are isomorphic for every complex space  $X$  [8, p. 161].

The following characterization of normal irreducible Stein spaces  $X$  by their global function algebra  $\mathcal{O}(X)$  is essentially contained in [2, § 1, p. 35].

**THEOREM 1.** *An irreducible Stein space  $X$  is normal if and only if the integral domain  $\mathcal{O}(X)$  is normal.*

An analysis of the proof shows that even when  $X$  is just irreducible and normal,  $\mathcal{O}(X)$  is also normal. Theorem 1 implies

**COROLLARY 1.** *For an irreducible Stein space  $X$  with normalization  $\tilde{X}$ , the integral closure  $\widetilde{\mathcal{O}(X)}$  of  $\mathcal{O}(X)$  is contained in  $\mathcal{O}(\tilde{X})$ .*

The following example shows that there are functions  $f \in \mathcal{O}(\tilde{X})$  which are not integral over  $\mathcal{O}(X)$ . In this example,  $X := (\mathbb{C}, \mathcal{O})$  is an irreducible

and locally irreducible Stein space given by a substructure of the canonical complex plane  $(\mathbf{C}, \mathcal{O})$ , which is then the normalization  $\tilde{X}$  of  $X$ . The substructure is defined by a "Strukturausdünnung" (see [10]) which results by replacing the stalks  $\mathcal{O}_n, n \in \mathbf{N}$ , with the stalks of generalized Neil parabolas becoming steeper as  $n$  increases. More precisely, let  $(p_n)_{n \in \mathbf{N}}$  be a strictly increasing sequence of prime numbers. For every  $n \in \mathbf{N}$ ,

$$X_n := \{(x, y) \in \mathbf{C}^2 : x^{p_n} = y^{p_n+1}\}$$

is an irreducible, locally irreducible analytic subset of  $\mathbf{C}^2$  with the origin as the only singularity and with normalization

$$\pi_n: \mathbf{C} \rightarrow X_n, t \mapsto (t^{p_n+1}, t^{p_n}).$$

Let  $f \in \mathcal{O}(\mathbf{C})$  be the identity and denote by  $\mathcal{O}_{X_n}$  the canonical complex structure on  $X_n$ . The germ  $f_0 \in \mathcal{O}_0$  of  $f$  at the origin is integral over  $\mathcal{O}_{X_n,0}$  with respect to a polynomial of degree  $p_n$ , and  $p_n$  is the minimal degree of all such polynomials.

Now define  $X := (\mathbf{C}, \mathcal{O}')$  as a substructure of the canonical plane  $(\mathbf{C}, \mathcal{O})$  with stalks

$$\mathcal{O}'_x \cong \begin{cases} \mathcal{O}_x & , \quad x \notin \mathbf{N} \\ \mathcal{O}_{X_n,0} & , \quad x = n \in \mathbf{N} \end{cases}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}'_n & \rightarrow & \mathcal{O}_n \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{O}_{X_n,0} & \xrightarrow{\pi_n^*} & \mathcal{O}_0, \end{array}$$

where  $\mathcal{O}'_n \rightarrow \mathcal{O}_n$  is the map induced by the identity  $(\mathbf{C}, \mathcal{O}) \rightarrow (\mathbf{C}, \mathcal{O}')$  and  $\mathcal{O}_n \cong \mathcal{O}_0$  is determined by the translation  $\mathbf{C} \rightarrow \mathbf{C}, z \mapsto z - n$ .

The identity  $f \in \mathcal{O}(\mathbf{C})$  is not integral over  $\mathcal{O}'(\mathbf{C})$ , because otherwise every polynomial of integral dependence would have degree at least  $p_n$  for all  $n \in \mathbf{N}$ .

In conclusion it should be mentioned that  $\mathcal{O}(\tilde{X})$  is almost integral over  $\mathcal{O}(X)$  [7, § 3] for every irreducible Stein space  $X$ , since  $X$  has a global universal denominator [10, E.73a].