5. Proof of the Fundamental Constraint

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.04.2024**

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Let $(p(t), v(t))$ be a curve in the unit tangent bundle US^3, such that $p(t)$ traces out a spherical helix in S^3 at constant speed, while $v(t)$ has constant coefficients with respect to the moving Frenet frame along this helix. We saw in section 1 that a geodesic in the unit tangent bundle must have this form, and also noted there that it will be sufficient to restrict our attention to the 3-sphere S^3.

In this section we will verify the Fundamental Constraint: $(p(t), v(t))$ is a geodesic in US^3 if and only if its slope equals the writhe of the helix $p(t)$. We will assume that the helix has nonzero curvature, and leave the degenerate case, in which $p(t)$ is a point or a great circle, until the very end.

The key step in the argument may be described as follows. Consider the 3-dimensional linear space of vector fields $aT(t) + bN(t) + cB(t)$ which can be written as constant coefficient combinations of the Frenet vectors along the helix $p(t)$. Covariant differentiation along the helix provides an endomorphism of this space, whose action was described in section 3. If we fix the value of t, this space becomes the tangent space to S^3 at $p(t)$. Here we may consider the action of the Riemann curvature transformation $R(v', v)$. The key step will be to compare these two endomorphisms.

In carrying out the argument, we will be blending Sasaki's two equations:

1) $v'' = -\langle v', v'\rangle\, v$

2) $p'' = R(v', v)p'$

with the three Frenet equations for the helix:

3) $T' = \kappa N$

4) $N' = -\kappa T - \tau B$

5) $B' = \tau N$.

To begin, assume that $(p(t), v(t))$ is a geodesic in US^3. For convenience, let t be an arc length parameter along $p(t)$. We first aim to show that the action of covariant differentiation coincides with that of the Riemann curvature transformation $R(v', v)$. To do this, we must verify

6) $T' = R(v', v)T$

7) $N' = R(v', v)N$

8) $B' = R(v', v)B$.

The unit tangent vector field $T(t) = p'(t)$, since t was set as an arc length parameter along $p(t)$. Making this substitution in Sasaki's equation 2) gives equation 6).

To get equation 7), combine equations 3) and 6) to get

9) $\kappa N = R(v', v)T$.

Then take covariant derivatives on both sides of this equation:

$$\kappa N' = R(v'', v)T + R(v', v')T + R(v', v)T' .$$

Sasaki's equation 1) and skew symmetry of R show that $R(v'', v) = 0$. Skew-symmetry alone gives $R(v', v') = 0$. In the third term on the right, replace T' by κN. Divide through by κ to get equation 7).

Covariant differentiation and the Riemann curvature transformation $R(v', v)$ are both skew symmetric endomorphisms of our 3-dimensional linear space. Equations 6) and 7) tell us that they agree on two out of the three basis vectors. Automatically, they must agree on the third, giving equation 8). Thus the two endomorphisms coincide.

From this, we want to conclude that slope $=$ writhe.

We've already described the action of covariant differentiation in section 3: it kills the instantaneous axis vector $U = \tau T - \kappa B$ and takes the orthogonal 2-plane to itself by a 90° rotation, followed by multiplication by the writhe.

Since we are on S^3, one can show that the Riemann curvature transformation $R(v', v)$ consists of orthogonal projection of the tangent 3-space onto the 2-plane spanned by v and v', followed by rotation by 90° in the direction from v to v', followed by multiplication by $|v'|$.

Since these two transformations coincide, writhe $= |v'|$. All this assumes that $|p'| = 1$. In general, we get

$$\text{writhe} = |v'| / |p'| = \text{slope},$$

verifying the necessity of the Fundamental Constraint.

Note also that, because the two transformations coincide, the vector $v(t)$ must be orthogonal to the instantaneous axis vector $U(t)$ of the helix $p(t)$, thus verifying the necessity of the Fundamental Constraint in its second formulation.

Conversely, suppose $(p(t), v(t))$ is a curve in US^3, with $p(t)$ tracing out a spherical helix in S^3 at constant speed, and $v(t)$ having constant coefficients with respect to the moving Frenet frame along this helix. In particular, $|v'(t)|$ is constant, and hence so is the slope $|v'(t)| / |p'(t)|$. Suppose this slope equals the writhe of the helix. We must show that $(p(t), v(t))$ is a geodesic in US^3.

As in the first part of the proof, we aim to show that the action of covariant differentiation coincides with that of the Riemann curvature transformation $R(v', v)$.

To this end, adjust the speed so that t is an arc length parameter along the helix $p(t)$. Hence $| v' |$ = writhe. But this is the maximum magnification of covariant differentiation, and can only be achieved when $v(t)$ is orthogonal to the instantaneous axis vector $U(t)$. Thus $<v, U> = 0$. Differentiate this equation, keeping in mind that $U' = 0$, and get $<v', U> = 0$. Hence v' is also orthogonal to the instantaneous axis.

But this means that the kernel and image of covariant differentiation coincide with the kernel and image of the Riemann curvature transformation $R(v', v)$. Since writhe $= | v' |$, the maximum magnifications of these two transformations also coincide. Then, by their special nature, so must the transformations themselves.

With this done, we can now check that $(p(t), v(t))$ is a geodesic in US^3 by verifying Sasaki's two equations.

Consider the vector field v''. Since covariant differentiation coincides with application of $R(v', v)$, the vector v'' is obtained from v by twice rotating the vv' plane by 90° and twice multiplying by $| v' |$. That is,

$$v'' = - <v', v'> v \, ,$$

which is Sasaki's first equation.

Next look at the vector field T'. This must be the same as $R(v', v)T$. But $T(t) = p'(t)$ and $T'(t) = p''(t)$, so we get

$$p'' = R(v', v)p' \, ,$$

which is Sasaki's second equation.

Hence $(p(t), v(t))$ must be a geodesic in US^3 by Sasaki's theorem, verifying the sufficiency of the Fundamental Constraint.

To verify the sufficiency of the Fundamental Constraint in its second formulation, suppose we begin instead with the information that $v(t)$ is orthogonal to the instantaneous axis vector $U(t)$. It is here that covariant differentiation achieves its maximum magnification, equal to the writhe of the helix $p(t)$. Thus $| v'(t) |$ = writhe. The above proof of sufficiency now applies, and we conclude again that $(p(t), v(t))$ must be a geodesic in US^3.

We complete the proof of the Fundamental Constraint by checking the two degenerate cases, again using Sasaki's equations.

If $p(t)$ is a constant point, then Sasaki's second equation is certainly satisfied, while the first tells us that $(p(t), v(t))$ is a geodesic in US^3 if and only if $v(t)$ traces out, at constant speed, a great circle in the tangent space to S^3 at that point.

If $p(t)$ is a great circle in S^3, travelled at constant speed, then $p'' = 0$, so Sasaki's second equation reads

$$R(v', v)p' = 0 .$$

This can be satisfied in two ways.

One is that $v' = 0$, so that $v(t)$ is a parallel vector field along $p(t)$. In this case, Sasaki's first equation is automatically satisfied, so $(p(t), v(t))$ must be a geodesic in US^3.

The other way for Sasaki's second equation to be satisfied is that v and v' are both orthogonal to p'. Parallel translate $v(t)$ backwards along $p(t)$ to the vector field $u(t)$ in the tangent space to S^3 at $p(0)$. Then Sasaki's first equation says that $u(t)$ traces out, at constant speed, a great circle orthogonal to $p'(0)$. Equivalently, $v(t)$ spins at constant but arbitrary speed along a great circle orthogonal to that of $p(t)$. In these circumstances, the curve $(p(t), v(t))$ will be a geodesic in US^3.

But these are precisely the interpretations of the Fundamental Constraint which were set in the introduction, and the proof is complete.

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(Reçu le 20 octobre 1987)

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