Objekttyp: **Group**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **28.03.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

The group of all invertible $n \times n$ upper triangular matrices will be denoted by B_n. Its subgroup consisting of all diagonal matrices is denoted by D_n. We have $B_n = U_n \rtimes D_n$ where U_n is the closed connected subgroup of B_n consisting of all unipotent elements of B_n.

We start with some preliminary facts.

THEOREM 1 (Lie-Kolchin). *Every connected solvable affine algebraic group can be embedded in some B_n as a closed subgroup.* □

COROLLARY. *If G is a connected solvable affine group then $G' \subset G_u$.* □

THEOREM 2 (Chevalley). *If N is a closed normal subgroup of an affine group G then there exists a homomorphism $f: G \to GL_n(k)$ such that* $\operatorname{Ker} f = N$. □

For the proofs of Theorems 1 and 2 see, for instance, [5, Theorems 6.7 and 5.1.3].

LEMMA 1. *If $f: G \to H$ is a surjective homomorphism of affine algebraic groups and $N := \operatorname{Ker} f$ then:*

(i) $f(G^0) = H^0$;

(ii) $f(G_u) = H_u$ *and* $f(G_s) = H_s$;

(iii) $\dim G = \dim N + \dim H$;

(iv) *If N and H are connected then G is connected.*

Proof. For the proofs of (i) and (iii) see for instance [4, Section 7.4]. (ii) follows from the fact that f preserves the Jordan decomposition [4, Theorem 2.4.8]. We shall sketch the proof of (iv). Since N is connected, we have $N \subset G^0$. By (i) we have $f(G^0) = H^0 = H$, and consequently $G = NG^0 = G^0$. □

We need a lemma to prove the centralizer theorem. For a more general version of this lemma see [2, Proposition (9.3)].

LEMMA 2. *Let N be a closed normal connected abelian unipotent subgroup of an affine group G and let $s \in G_s$. Then $M := \{sus^{-1}u^{-1} : u \in N\}$ is a closed connected subgroup of N, the multiplication map $\mu: M \times Z_N(s) \to N$ is bijective, and $Z_N(s)$ is connected.*

Proof. Since N is abelian, the map $f: N \to N$, defined by $f(u) = sus^{-1}u^{-1}$, is a morphism of algebraic groups whose kernel is $Z_N(s)$ and image M, so M is a closed connected subgroup of N. If $x \in M \cap Z_N(s)$ then $x = sus^{-1}u^{-1}$ for some $u \in N$. Since $usu^{-1} = x^{-1}s = sx^{-1}$ is semisimple and x is unipotent, the uniqueness of the Jordan decomposition implies that $x = 1$. Hence $M \cap Z_N(s) = 1$ and so μ is injective. By Lemma 1 (iii) we have $\dim N = \dim M + \dim Z_N(s)$, which implies that the homomorphism μ is also surjective, i.e., $MZ_N(s) = N$. The same argument shows that $MZ_N(s)^0 = N$, and so $Z_N(s)$ must be connected. □

THEOREM 3. *If G is a connected solvable affine group and $s \in G_s$ then $Z_G(s)$ is connected and $G = G_u Z_G(s)$.*

Proof. We use induction on $\dim G$. If G is abelian the assertions are trivial. Otherwise let N be the last non-trivial term of the derived series of G. By the Corollary of Theorem 1, N is unipotent. We now apply Theorem 2 to this G and N. Let f be as in that theorem. We shall write $\bar{x}$ for $f(x)$ and $\bar{G}$ for $f(G)$.

Let $z \in G$ be such that $\bar{z} \in Z_{\bar{G}}(\bar{s})$. Then $szs^{-1}z^{-1} \in N$. By Lemma 2 there exists $u \in N$ and $v \in Z_N(s)$ such that $szs^{-1}z^{-1} = sus^{-1}u^{-1} \cdot v$. Since v commutes with u and s, and $zsz^{-1} = v^{-1} \cdot usu^{-1}$, it follows that $v = 1$. Thus $u^{-1}z \in Z_G(s)$ and consequently we have a short exact sequence

$$1 \to Z_N(s) \hookrightarrow Z_G(s) \to Z_{\bar{G}}(\bar{s}) \to 1 .$$

By Lemma 2, $Z_N(s)$ is connected. By Lemma 1 (iii) we have $\dim \bar{G} < \dim G$. By induction hypothesis, we conclude that $Z_{\bar{G}}(\bar{s})$ is connected and that $\bar{G} = (\bar{G})_u \cdot Z_{\bar{G}}(\bar{s})$. Now Lemma 1 (iv) implies that $Z_G(s)$ is connected. By part (ii) of the same lemma we have $f(G_u) = (\bar{G})_u$ and so $f(G_u Z_G(s)) = (\bar{G})_u Z_{\bar{G}}(\bar{s}) = \bar{G}$. Since $N \subset G_u$, it follows that $G = G_u Z_G(s)$. □

We now proceed to prove the main results about the structure of connected solvable affine groups. But first we need two lemmas.

LEMMA 3. *Let $S \subset B_n$ be a commuting set of semisimple elements. Then there exists $b \in B_n$ such that $b^{-1}Sb \subset D_n$.*

Proof. It is an elementary fact of linear algebra that there exists $a \in GL_n(k)$ such that $a^{-1}Sa \subset D_n$. Hence if $M_n(k)$ is the algebra of n by n matrices over k and A its subalgebra generated by S, we know that A is semisimple (and commutative). Let $V := k^n$ be the space of column

vectors and let $e_1, ..., e_n$ be its standard basis. We shall view V as a left $M_n(k)$-module via matrix multiplication. The subspace V_i spanned by the vectors $e_1, ..., e_i$ is an A-submodule of V for each i. Since A is semisimple, there exist $v_i \in V_i \backslash V_{i-1}$, $1 \leqslant i \leqslant n$, such that $Av_i = kv_i$. Thus if b is the matrix whose i-th column is v_i, $1 \leqslant i \leqslant n$, then $b \in B_n$ and $b^{-1}Sb \subset D_n$. □

LEMMA 4. *If G is a connected solvable affine group, $T \subset G_s$ a closed subgroup, and $G = G_u T$ then T is a torus and $G = G_u \rtimes T$.*

Proof. By the Lie-Kolchin theorem we may assume that G is a closed subgroup of some B_n. By using the projection map $B_n \to D_n$ we obtain a short exact sequence $1 \to G_u \hookrightarrow G \xrightarrow{p} D \to 1$, where $D \subset D_n$ is a torus. Since $D = p(G) = p(G_u T) = p(T)$, Lemma 1 (i) implies that $p(T^0) = D$. Thus $G = G_u T^0$ and using $T \cap G_u = 1$ we conclude that $T = T^0$. In particular T is abelian and by Lemma 3 we may assume that $T \subset D_n$, i.e., $T = D$. Since $B_n = U_n \rtimes D_n$, $G_u \subset U_n$, $T = D \subset D_n$, and $G = G_u T$, it follows that $G = G_u \rtimes T$. □

THEOREM 4. *Let G be a connected solvable affine group. Then $G = G_u \rtimes T$ where T is a maximal torus. In particular, G_u is connected.*

Proof. We use induction on $\dim G$. Assume first that $G_s \subset Z(G)$. Then $G_s = Z(G)_s$ is a closed subgroup of G and $G = G_u G_s$. The assertion then follows from Lemma 4. Now assume that there exists $s \in G_s \backslash Z(G)$. Then $Z_G(s)$ is a proper closed subgroup of G, see e.g. [4, Section 8.2]. By Theorem 3 it is connected and $G = G_u Z_G(s)$. By induction hypothesis there exists a torus T such that $Z_G(s) = Z_G(s)_u T$. Then $G = G_u Z_G(s) = G_u T$ and $G = G_u \rtimes T$ by Lemma 4. □

THEOREM 5. *Let $G = G_u \rtimes T$ be a connected solvable affine group. Then every $s \in G_s$ is conjugate to an element of T.*

Proof. We use induction on $\dim G$. We have $s = ut$ where $u \in G_u$ and $t \in T$. If G is abelian then $u = 1$ and $s = t$. Otherwise let N be the last non-trivial term of the derived series of G. By the corollary of Theorem 1 we have $N \subset G_u$. Hence N is a closed connected normal abelian unipotent subgroup of G. By Theorem 2 and the induction hypothesis there exists $x \in G$ such that $xsx^{-1} = tv$ where $v \in N$. By Lemma 2, $v = t^{-1}utu^{-1}z$ where $u \in N$ and $z \in Z_N(t)$. Hence $xsx^{-1} = utu^{-1}z$. Since $xsx^{-1}, utu^{-1} \in G_s$, $z \in G_u$, and z commutes with u and t, it follows that $z = 1$ and consequently $xsx^{-1} = utu^{-1}$. □