

# K NON-DYADIC

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

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## K NON-DYADIC

We begin with a well known lemma, valid for an arbitrary field of characteristic 0.

LEMMA. Let  $K_0$  be a subfield of  $K$  which contains  $\mathbf{Q}^c \cap K$ . Then

$$S(K) = K \otimes S(K_0)$$

where  $K \otimes S(K_0)$  denotes the subgroup of  $B(K)$  obtained from  $S(K_0)$  by extension of scalars.

*Proof.* See Prop. 4.6 in [Y]; a proof of a more general result can be found in [M]. Since the proof is short, we give it here. We can assume  $K_0 = \mathbf{Q}^c \cap K$ . Let  $\beta \in S(K)$  and let  $A$  be a Schur algebra in  $\beta$ , i.e. a simple component of some group algebra  $KG$  with Brauer class  $[A] = \beta$ . Then  $A$  is also a direct summand of  $K \otimes A_0$  for some simple component  $A_0$  of  $K_0G$ . But the center of  $A_0$  is a sub-cyclotomic extension of  $K_0$  (see exercise 9.15, [I], e.g.), so is  $K_0$  since it is also contained in  $K$ . Thus  $[A_0] \in S(K_0)$  and  $A = K \otimes A_0$ . It follows that  $S(K) \subseteq K \otimes S(K_0)$  and the reverse inclusion is obvious.  $\square$

This lemma allows us to assume, from now on, that  $K$  is a sub-cyclotomic extension of  $\mathbf{Q}_p$ , i.e. a (finite) abelian extension of  $\mathbf{Q}_p$ .

We shall denote the group of roots of unity of a field  $L$  by  $\mu(L)$ . The subgroup of roots of unity of order a power of  $p$ , resp. of order relatively prime to  $p$ , is denoted by  $\mu(L)_p$  resp.  $\mu(L)_{p'}$ . The group of all roots of unity, i.e.  $\mu(\mathbf{Q}_p^c)$ , will be denoted by  $\mu$ , with  $\mu_2$  and  $\mu_2'$  having the obvious meanings.

Assume now that  $p$  is odd. Since  $\mu(\mathbf{Q}_p)$  is  $\cong \mathbf{Z}/p-1$ , the root of unity theorem of Benard and Schacher (Th. 6.1, [Y]) and the fact that  $B(\mathbf{Q}_p) \cong \mathbf{Q}/\mathbf{Z}$  (see [S], e.g.) imply that  $S(\mathbf{Q}_p) \hookrightarrow \mathbf{Z}/p-1$ . (In fact this map is an isomorphism). By the theory of central simple algebras over a local field, we can identify  $B(K)$  with  $H^2(\mathcal{G}(\mathbf{Q}_p^c/K), (\mathbf{Q}_p^c)^*)$ , (which we also denote by  $H^2(\mathbf{Q}_p^c/K)$ ). By the Brauer-Witt theorem (Cor. 3.11, [Y]),  $S(K)$  is thereby identified with the image of the canonical map

$$H^2(\mathcal{G}(\mathbf{Q}_p^c/K), \mu) \rightarrow H^2(\mathcal{G}(\mathbf{Q}_p^c/K), (\mathbf{Q}_p^c)^*), \quad (\mu = \mu(\mathbf{Q}_p^c)),$$

which we denote by  $H_c^2(\mathbf{Q}_p^c/K)$ . The (cohomological) corestriction map  $B(K) \rightarrow B(\mathbf{Q}_p)$  carries  $S(K)$  into  $S(\mathbf{Q}_p)$  since, on the cocycle level, it takes a cocycle  $f$  to a cocycle whose values are products of the values of  $f$  (see [W], e.g.). Furthermore the corestriction is injective in this case

(p. 175, [S]) and so  $S(K)$  is finite — in fact it is a subgroup of  $\mathbf{Z}/p-1$ . We may therefore choose a primitive  $m^{\text{th}}$  root of unity  $\varepsilon_m$  so that

$$S(K) = H_c^2(\mathbf{Q}_p(\varepsilon_m)/K).$$

We may also assume that  $p \mid m$ , i.e. that  $\varepsilon_p \in \mathbf{Q}_p(\varepsilon_m)$ .

We now show that  $\mathbf{Q}_p(\varepsilon_m)$  can be replaced by a field  $L$  so that  $L/K$  is cyclic and totally tamely ramified. First of all, by Lemma 4.1, [Y], we can assume that  $\mathbf{Q}_p(\varepsilon_m)$  is the (disjoint) compositum  $UV$  of an unramified extension  $U/K$  and a totally ramified extension  $V/K$ . Since the order of  $S(K)$  is relatively prime to  $p$ ,  $S(K)$  is the image  $H_c^2(\mathbf{Q}_p(\varepsilon_m)/K)'$  of the canonical map

$$H^2(\mathcal{G}(\mathbf{Q}_p(\varepsilon_m)/K), \mu(\mathbf{Q}_p(\varepsilon_m))_{p'}) \rightarrow H^2(\mathbf{Q}_p(\varepsilon_m)/K).$$

Since  $UV/V$  is unramified,  $N_{UV/V}(\mu(UV)_{p'}) = \mu(V)_{p'}$ , and it follows from the inflation-restriction sequence (see Lemme 1, [F]) that the inflation

$$H^2(\mathcal{G}(V/K), \mu(V)_{p'}) \rightarrow H^2(\mathcal{G}(UV/K), \mu(UV)_{p'})$$

is an isomorphism (since  $UV/V$  cyclic implies that  $H^2(\mathcal{G}(UV/V), \mu(UV)_{p'}) \cong H^0(\mathcal{G}(UV/V), \mu(UV)_{p'}) = 1$ ). Thus  $S(K) = H_c^2(V/K)'$ . Let  $L/K$  be the tamely ramified part of  $V/K$ . Since the  $p'$  roots of unity in a local field are the same as the non-zero elements in the residue class field,  $\mu(V)_{p'} = \mu(L)_{p'} = \mu(K)_{p'}$  and so  $N_{V/L}(\mu(V)_{p'}) = \mu(L)_{p'}$  because  $(V:L)$  is a power of  $p$ . Once again the inflation-restriction sequence shows that  $S(K) = H_c^2(L/K)'$ .

Consider now the cup product pairing

$$(1) \quad \smile : \mathcal{G}(L/K) \times H^2(L/K) \rightarrow K^*/N_{L/K}L^*.$$

See for example pp. 139-140, [C-F]. It is known that there is a “canonical class”  $u_{L/K}$  in  $H^2(L/K)$  with the property that the map  $\sigma \mapsto \sigma \smile u_{L/K}$  is an isomorphism  $\mathcal{G}(L/K) \rightarrow K^*/N_{L/K}L^*$ . It follows that if  $\sigma$  is a generator of  $\mathcal{G}(L/K)$ , the map

$$\sigma \smile : H^2(L/K) \rightarrow K^*/N_{L/K}L^*$$

is also an isomorphism. We wish to identify the image of  $H_c^2(L/K)'$  under this map. The cohomology class  $[f]$  of the cocycle  $f$  has image  $\prod_{\tau} f(\tau, \sigma) \bmod N_{L/K}L^*$  (Lemme 4, p. 186, [S]). Since  $\mathcal{G}(L/K)$  is cyclic with generator  $\sigma$ , every cohomology class with coefficients in an arbitrary  $\mathcal{G}(L/K)$ -module  $A$  is represented by a cocycle  $f$  of the form

$$f(\sigma^i, \sigma^j) = \begin{cases} 1 & \text{if } i+j < d, \\ a & \text{if } i+j \geq d. \end{cases}$$

Here  $d = |\mathcal{G}(L/K)|$ ,  $0 \leq i, j < d$ , and  $a$  is an arbitrary element of  $A^{\mathcal{G}(L/K)}$ . If  $A = L^*$ , it follows that a class in  $H^2(L/K)$  is in  $H_c^2(L/K)'$  iff it contains such an  $f$  with  $a \in \mu(K)_{p'}$ . Since it is clear that  $\sigma \smile [f] = a \bmod N_{L/K}L^*$ , we see that the image of  $H_c^2(L/K)'$  is

$$\mu(K)_{p'} N_{L/K}L^*/N_{L/K}L^* \cong \mu(K)_{p'}/\mu(K)_{p'} \cap N_{L/K}L^*.$$

But it is easy to see that  $\mu(K)_{p'} \cap N_{L/K}L^* = N_{L/K}\mu(L)_{p'}$ , so we have an isomorphism

$$S(K) = \mu(K)_{p'}/N_{L/K}\mu(L)_{p'}$$

depending only on the choice of  $\sigma$ .

We now show that the norm residue symbol

$$v_K = ( \quad , \mathbf{Q}_p^c/K ): K^* \rightarrow \mathcal{G}(\mathbf{Q}_p^c/K)$$

induces an isomorphism of  $\mu(K)_{p'}/N_{L/K}\mu(L)_{p'}$  onto  $\text{tor } \mathcal{G}(\mathbf{Q}_p^c/K)$ . It is clear that the image of  $\mu(K)_{p'}$  is contained in  $\text{tor } \mathcal{G}(\mathbf{Q}_p^c/K)$ . Let  $v = ( \quad , \mathbf{Q}_p^c/\mathbf{Q}_p )$ . The diagram

$$\begin{array}{ccc} K^* & \xrightarrow{v_K} & \mathcal{G}(\mathbf{Q}_p^c/K) \\ N_{K/\mathbf{Q}_p} \downarrow & & \downarrow \text{incl.} \\ \mathbf{Q}_p^* & \xrightarrow{v} & \mathcal{G}(\mathbf{Q}_p^c/\mathbf{Q}_p) \end{array}$$

is commutative (Prop. 10, ch. XIII, [S]). Recall now that  $\varepsilon_p \in \mathbf{Q}_p(\varepsilon_m)$ . It follows that the tame ramification index of  $L/\mathbf{Q}_p$  is  $p-1$ . Therefore if  $L'/\mathbf{Q}_p$  is the maximal unramified subextension of  $L/\mathbf{Q}_p$ , then  $(L:L')$  is a  $p$ -power multiple of  $p-1$ . Since  $\mu(L)_{p'} = \mu(L')_{p'}$  and  $N_{L'/\mathbf{Q}_p}\mu(L')_{p'} = \mu(\mathbf{Q}_p)_{p'} \cong \mathbf{Z}/p-1$ , the kernel  $\kappa$  of the restriction of  $N_{K/\mathbf{Q}_p}$  to  $\mu(K)_{p'}$  is  $\subseteq N_{L/K}\mu(L)_{p'}$ . On the other hand if one factors  $N_{K/\mathbf{Q}_p}$  through the tame and unramified closures of  $\mathbf{Q}_p$  in  $K$ , one sees that  $N_{L/K}$  on  $\mu(K)_{p'}$  is  $\varepsilon \mapsto \varepsilon^{e(p^f-1)/(p-1)}$  where  $e$  and  $f$  are resp. the ramification and inertial indices of  $K/\mathbf{Q}_p$ . It follows that  $\kappa = N_{L/K}\mu(L)_{p'}$  which is equal to  $\ker v_K \mid_{\mu(K)_{p'}}$ , since  $v$  is injective.

Now  $v$  maps the torsion subgroup  $\mu(\mathbf{Q}_p) \cong \mathbf{Z}/p-1$  of  $\mathbf{Q}_p^*$  onto the torsion subgroup of  $\mathcal{G}(\mathbf{Q}_p^c/\mathbf{Q}_p)$ . Furthermore an element  $a \in \mathbf{Q}_p^*$  is mapped into  $\mathcal{G}(\mathbf{Q}_p^c/K)$  iff  $a \in N_{K/\mathbf{Q}_p}K^*$ . It follows at once that  $v$  maps  $N_{K/\mathbf{Q}_p}\mu(K)_{p'}$  isomorphically onto  $\text{tor } \mathcal{G}(\mathbf{Q}_p^c/K)$ . This proves our main theorem in the case  $p$  odd.

## K DYADIC

It would be very nice to have a unified proof for the dyadic and non-dyadic cases along the lines of the one above for the non-dyadic case. However that would require a "deflation"  $H_c^2(V/K) \cong H_c^2(L/K)$  to some cyclic extension  $L/K$  in order that the cup product pairing (1) be non-degenerate on both sides. U. Jannsen has shown that this is impossible in general. Since  $H^2(L_1/K) = H^2(L_2/K)$  (when inflated to a common extension) if  $(L_1:K) = (L_2:K)$ , one can try to replace the cyclotomic extension  $V/K$  by a some cyclic but possibly non-cyclotomic extension to achieve non-degeneracy. This is done, however, at the expense of losing the identification of  $S(K)$  as the subgroup of cyclotomic cocycles. This is essentially what is done in the second half of the following proof.

Since  $\mu(\mathbf{Q}_2) = \pm 1$ , it follows (as in the non-dyadic case) that  $S(K)$  is 1 or  $\pm 1$ . Thus to prove the theorem it suffices to show that

$$(2) \quad S(K) \neq 1 \Leftrightarrow -1 \in \mathcal{G}(\mathbf{Q}_2^c/K).$$

Before beginning we recall a few facts about Galois groups of  $\mathbf{Q}_2^c$ . Let  $\varepsilon_m$  be a primitive  $m^{\text{th}}$  root of unity and write  $m = 2^n m'$  where  $m'$  is odd. Let  $f$  be the smallest integer such that  $m' \mid 2^f - 1$ . Then if  $n \geq 2$ ,

$$\mathcal{G}(\mathbf{Q}_2(\varepsilon_m)/\mathbf{Q}_2) \cong \mathbf{Z}/f \times (\mathbf{Z}/2^n)^* \cong \mathbf{Z}/f \times \mathbf{Z}/2^{n-2} \times \mathbf{Z}/2.$$

Taking  $\varprojlim$  over  $m$  one gets

$$(3) \quad \mathcal{G}(\mathbf{Q}_2^c/\mathbf{Q}_2) \cong \hat{\mathbf{Z}} \times \hat{\mathbf{Z}}_2 \times \mathbf{Z}/2.$$

The topological generator 1 of  $\hat{\mathbf{Z}}$  is the Frobenius of the maximal unramified extension  $\mathbf{Q}_2(\mu_{2'})$  of  $\mathbf{Q}_2$ . The topological generator 1 of  $\hat{\mathbf{Z}}_2$  and the generator 1 of  $\mathbf{Z}/2$  are the automorphisms of the field  $\mathbf{Q}_2(\mu_2)$  determined by  $\varepsilon \mapsto \varepsilon^5$  and  $\varepsilon \mapsto \varepsilon^{-1}$  resp. for all  $\varepsilon \in \mu_2$  (see e.g. [H], § 4, 5). We shall denote these automorphisms by  $\sigma_5$  and  $\sigma_{-1}$  resp.

From (3) we get a "primary decomposition"

$$(4) \quad \mathcal{G}(\mathbf{Q}_2^c/\mathbf{Q}_2) \cong \prod_{p \neq 2} \hat{\mathbf{Z}}_p \times (\hat{\mathbf{Z}}_2 \times \hat{\mathbf{Z}}_2 \times \mathbf{Z}/2)$$

since  $\hat{\mathbf{Z}} \cong \prod \hat{\mathbf{Z}}_p$ . Since  $\mathcal{G}(\mathbf{Q}_2^c/K)$  is an open subgroup, one can show that the isomorphism implied in (4) restricts to an isomorphism

$$\mathcal{G}(\mathbf{Q}_2^c/K) \cong \prod_{p \neq 2} k_p \hat{\mathbf{Z}}_p \times C_K \cong \hat{\mathbf{Z}} \times \hat{\mathbf{Z}}_2 \times D_K$$

where  $C_K$  is a  $\hat{\mathbf{Z}}_2$ -submodule of finite index in the component  $\hat{\mathbf{Z}}_2 \times \hat{\mathbf{Z}}_2 \times \mathbf{Z}/2$  of (4),  $k_p$  is an integer (or a power of  $p$ ) = to 1 for almost all  $p$ , and  $D_K$  is either the trivial group or  $\langle \sigma_{-1} \rangle$ .

We now begin the proof of (2). Suppose first that  $\sigma_{-1} \notin \mathcal{G}(\mathbf{Q}_2^c/K)$ , i.e. that  $\mathcal{G}(\mathbf{Q}_2^c/K) \cong \hat{\mathbf{Z}} \times \hat{\mathbf{Z}}_2$ . It suffices to show that  $H^2(\mathcal{G}(\mathbf{Q}_2^c/K), \mu_2)$  is trivial. Let  $K_{nr}$  be the unramified closure of  $K$  in  $\mathbf{Q}_2^c$ . Then  $K_{nr} = K(\mu_{2^r})$  and  $\mathcal{G}(K_{nr}/K) \cong \hat{\mathbf{Z}}$ .

Let  $C_n$  denote the cyclic group of order  $n$ .

LEMMA. Suppose  $C_{2^k}$  operates faithfully on  $C_{2^h}$ . Then  $H^n(C_{2^k}, C_{2^h}) = 1$  for all  $n \geq 1$  except in one case:  $k = 1$  and the non-trivial automorphism in  $C_2$  inverts the elements of  $C_{2^h}$  (i.e. " $C_{2^k} = \langle \sigma_{-1} \rangle$ ").

This is a well-documented fact, although perhaps not exactly in this form (see e.g. [N], 4.8, or the proof of Lemma 2, [L]). By the Herbrand theory for the cohomology of cyclic groups (see e.g. [S], ch. VIII, § 4), it suffices to show that  $\hat{H}^0(C_{2^k}, C_{2^h}) = 1$ , i.e. that every fixed element is a norm. There is generator of  $C_{2^k}$  which acts by raising the elements of  $C_{2^h}$  to either the power  $5^{2^{h-k-2}}$ , or possibly the power  $-5$  if  $k = h - 2$  (again [H], § 4, 5). Then a straightforward calculation leads to the desired result (one uses the fact that  $2^{r+2} \parallel (5^{2^r} - 1)$  for all  $r \geq 0$ ).  $\square$

Since

$$\mathbf{Q}_2^c = K_{nr}(\mu_2), H^n(\mathcal{G}(\mathbf{Q}_2^c/K_{nr}), \mu_2) = \varinjlim H^n(\mathcal{G}(L/K_{nr}), \mu(L)_2)$$

where  $L$  runs over the fields  $K_{nr}(\varepsilon_{2^h})$ , and so is trivial by the lemma for  $n \geq 1$ . Thus the inflation-restriction sequence (p. 126, [C-F])

$$1 \rightarrow H^2(\mathcal{G}(K_{nr}/K), \mu(K_{nr})_2) \rightarrow H^2(\mathcal{G}(\mathbf{Q}_2^c/K), \mu_2) \rightarrow H^2(\mathcal{G}(\mathbf{Q}_2^c/K_{nr}), \mu_2) = 1$$

is exact whence the inflation is an isomorphism. But  $\mathcal{G}(K_{nr}/K) = \hat{\mathbf{Z}}$  has cohomological dimension 1, so  $H^2(\mathcal{G}(K_{nr}/K), \mu(K_{nr})_2)$  is 1, hence  $H^2(\mathcal{G}(\mathbf{Q}_2^c/K), \mu_2)$  is also 1 as desired. (I am grateful to U. Jannsen for the foregoing proof).

We now assume that  $\sigma_{-1} \in \mathcal{G}(\mathbf{Q}_2^c/K)$ . This part of the proof is derived from pp. 540-542 in [J] and pp. 467-468, [L]. (F. Lorenz has asked me to point out that the proof on pp. 465-466 of the latter paper is incomplete — one must show that  $\rho$  is the identity on  $k$ .)

LEMMA 1.  $K(\varepsilon_4)/K$  is ramified of degree 2.

*Proof.* It is clear that the extension is of degree 2. Suppose it is unramified. Let  $q$  be the number of elements in the residue class field of

$K(\varepsilon_4)$ . Then  $K(\varepsilon_4) = K(\varepsilon_{q-1})$ . But  $K(\varepsilon_{q-1})$  is left element-wise fixed by  $\sigma_{-1}$ , which contradicts the fact that  $\varepsilon_4$  is *not* left fixed.  $\square$

Let  $h$  be the smallest integer  $\geq 2$  such that there is an odd integer  $m$  with the property that  $L = \mathbf{Q}_2(\varepsilon_{2^h}, \varepsilon_m)$  contains  $K$ . By replacing  $m$  by a suitable multiple, we can suppose that the residue class degree of  $L/K$

$$f(L/K) \equiv 0 \pmod{2^h}.$$

Let  $\mathcal{G}$  be the Galois group of this extension. We shall construct a Schur class of  $K$  using  $L/K$ . For this we use the following very useful lemma. Let  $G$  be a finite abelian group, written as the direct sum of cyclic subgroups:

$$G = C_1 \oplus C_2 \oplus \dots \oplus C_r$$

where each  $C_i$  is of order  $c_i$  with generator  $\sigma_i$ . Let  $A$  be a  $G$ -module, written multiplicatively. Define the operators  $\Delta_{\sigma_i}$  and  $N_{\sigma_i}$  on  $A$  by

$$\Delta_{\sigma_i} a = a^{\sigma_i^{-1}}, N_{\sigma_i} a = a^{1 + \sigma_i + \sigma_i^2 + \dots + \sigma_i^{c_i - 1}}.$$

LEMMA. Let  $\gamma$  be a cohomology class in  $H^2(G, A)$ , and let  $f$  be a normalized cocycle in  $\gamma$ . Then the elements

$$\begin{aligned} a_i &= f(\sigma_i, \sigma_i) f(\sigma_i^2, \sigma_i) \dots f(\sigma_i^{c_i - 1}, \sigma_i), \\ a_{ij} &= f(\sigma_i, \sigma_j) / f(\sigma_j, \sigma_i) \quad (i \neq j) \end{aligned} \tag{5}$$

satisfy the following relations:

$$\Delta_{\sigma_i} a_j = \begin{cases} 1 & \text{if } i = j \\ N_{\sigma_j} a_{ij} & \text{if } i \neq j \end{cases}, \tag{6}$$

$$a_{ij} a_{ji} = 1 \quad (i \neq j), \quad \Delta_{\sigma_i} a_{jk} \cdot \Delta_{\sigma_j} a_{ki} \cdot \Delta_{\sigma_k} a_{ij} = 1 \quad (i, j, k \text{ distinct}).$$

Conversely if we have elements  $a_i$  and  $a_{ij}$  in  $A$  satisfying (6), then there is a uniquely determined cohomology class  $\gamma$  in  $H^2(G, A)$  and a normalized cocycle  $f$  in  $\gamma$  bearing the relationship (5) to the  $a_i$  and  $a_{ij}$ .

*Proof.* This is just a restatement of the abelian case of [Z], III, § 8, Theorem 22, in terms of cocycles. See also [Y], pp. 15-19.  $\square$

We now apply this to the situation at hand:  $G = \mathcal{G}$  and  $A = \mu(L)_2 = \langle \varepsilon_{2^h} \rangle$ . First of all we note that the restriction  $\sigma_1$  of  $\sigma_{-1}$  to  $L$  is a non-trivial element of  $\mathcal{G}$ , and that the minimality of  $h$  implies that

$K(\varepsilon_4, \varepsilon_m) = K(\varepsilon_{2^h}, \varepsilon_m) = L$  (see e.g. Lemma 3.3, [J]). Since  $K(\varepsilon_4)/K$  is ramified and  $K(\varepsilon_m)$  is unramified (because  $m$  is odd),  $\mathcal{G}$  is the direct product of the Galois groups  $\langle \sigma_1 \rangle$  of  $L/K(\varepsilon_m)$  and  $\langle \sigma_2 \rangle$  (say) of  $L/K(\varepsilon_4)$  of orders 2 and  $f$  respectively. We now choose  $a_1 = 1 = a_2$  and  $a_{12} = \varepsilon_{2^h} = a_{21}^{-1}$ . Then  $N_{\sigma_1} a_{21} = \varepsilon_{2^h}^{-1} \varepsilon_{2^h} = 1$  and  $N_{\sigma_2} a_{12} = \varepsilon_{2^h}^s$  where, if  $\sigma_2(\varepsilon_{2^h}) = \varepsilon_{2^h}^r$ ,

$$s = 1 + r + r^2 + \dots + r^{f-1}.$$

Since  $\sigma_2(\varepsilon_4) = \varepsilon_4$ , we have  $r \equiv 1 \pmod{4}$ .

Claim:  $s \equiv 0 \pmod{2^h}$ . If  $\sigma_2(\varepsilon_{2^h}) = \varepsilon_{2^h}$ , we choose  $r = 1$ ; then  $s = f \equiv 0 \pmod{2^h}$ .

Suppose then that  $\sigma_2(\varepsilon_{2^h}) \neq \varepsilon_{2^h}$ , and write  $s = (r^f - 1)/(r - 1)$ . Now  $r = 1 + 2^k a$  where  $h > k \geq 2$  and  $a$  is odd. By induction  $r^{2^i} = 1 + 2^{k+i} a_i$  ( $a_i$  an odd integer) for all  $i \geq 0$ , whence the claim.

It follows of course that  $N_{\sigma_2} a_{12} = 1$ . Therefore the above lemma provides a 2-cocycle  $f$  with coefficients in  $\langle \varepsilon_{2^h} \rangle$ . We now consider it to have coefficients in  $L^*$  and so its cohomology class  $\gamma = [f]$  is a Schur class in  $B(K)$ . We shall show that this class is non-trivial, which will finish the proof of the theorem. This will be effected by showing that  $\gamma$  is the inflation of a non-trivial Brauer class arising from the extension  $K(\varepsilon_m)/K$  — this latter class will not arise from a cyclotomic cocycle but this of course does not matter.

We shall use the crossed-product algebra  $A = (L/K, f)$  in order to carry this out. As a vector space over  $L$  it has a basis  $u_1^i u_2^j$  where  $0 \leq i < 2$  and  $0 \leq j < f$ , with  $u_1^2 = 1 = u_2^f$  and  $u_1 u_2 u_1^{-1} u_2^{-1} = \varepsilon_{2^h}$ . We replace  $u_2$  by  $u'_2 = \pi u_2$  where  $\pi = \varepsilon_4(1 - \varepsilon_{2^h})$ . The new parameters are

$$a'_1 = u_1^2 = 1, \quad a'_{12} = u_1 u'_2 u_1^{-1} u'_2^{-1} = 1, \quad a'_2 = u_2'^f = N_{\sigma_2} \pi.$$

By (6),  $\Delta_{\sigma_1} a'_2 = N_{\sigma_2} a'_{12} = 1$  and  $\Delta_{\sigma_2} a'_2 = 1$ , so  $N_{\sigma_2} \pi \in K$ . Since  $u_1$  and  $u'_2$  commute with each other, it follows easily that

$$A = (K(\varepsilon_4)/K, 1) \otimes (K(\varepsilon_m)/K, N_{\sigma_2} \pi).$$

The first of these crossed-product algebras is clearly split but the second is *not* split:  $\pi$  is a prime element of  $L$ , so  $N_{\sigma_2} \pi$  has order (valuation)  $f$  in  $K(\varepsilon_4)$  ( $L/K(\varepsilon_4)$  is unramified), hence order  $1/2 f$  in  $K$  ( $K(\varepsilon_4)/K$  is ramified); but the (non-zero) norms in  $K$  from  $K(\varepsilon_m)$  are exactly the elements whose order is a multiple of  $f$  (since the extension is unramified of degree  $f$ ). Thus  $A$  is not split, as desired.  $\square$