

2. Construction of $O(\tilde{X})$ from $O(X)$ for Stein spaces X

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

2. CONSTRUCTION OF $\mathcal{O}(\tilde{X})$ FROM $\mathcal{O}(X)$ FOR STEIN SPACES X

According to a theorem of Oka [12], the normalization sheaf $\tilde{\mathcal{O}}$ of weakly holomorphic functions on a complex space $(X, \mathcal{O})$ is coherent. Consequently, there is a canonical topology making $\tilde{\mathcal{O}}$ a Fréchet sheaf; the global weakly holomorphic functions $\tilde{\mathcal{O}}(X)$ will always carry this topology. Since the holomorphic functions $\mathcal{O}(\tilde{X})$ on the normalization $\tilde{X}$ of X are topologically isomorphic to $\tilde{\mathcal{O}}(X)$ [8, 8.3], the question posed in the introduction can now be answered.

MAIN THEOREM. *For an irreducible Stein space X , the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ is dense in $\tilde{\mathcal{O}}(X)$.*

Proof. Let $\pi: \tilde{X} \rightarrow X$ be the normalization of X and put $A := \widetilde{\mathcal{O}(X)}$. Since π is proper, $\tilde{X}$ is $\mathcal{O}(X)$ -convex and therefore $\bar{A}$ -convex. Note that Corollary 1 implies $A \subset \tilde{\mathcal{O}}(X)$ and that $\bar{A}$ is the closure of A with respect to the canonical topology in $\tilde{\mathcal{O}}(X)$.

Consider the equivalence relation R on $\tilde{X}$ defined by $\bar{A}$, i.e. $(x, y) \in R$ iff for every $f \in \bar{A}$, $f(x) = f(y)$. Rossi's theorem [13] ensures that the topological quotient $Y := \tilde{X}/R$ can be given the complex structure of a Stein space such that the projection $p: \tilde{X} \rightarrow Y$ is holomorphic and proper and the map $p^*: \mathcal{O}(Y) \rightarrow \mathcal{O}(\tilde{X})$, $f \mapsto f \circ p$, induces an isomorphism $\mathcal{O}(Y) \cong \bar{A}$.

It suffices to show that every $f \in \mathcal{O}(\tilde{X})$ can be factorized through a holomorphic function on Y , meaning that an $F \in \mathcal{O}(Y)$ exists with $F \circ p = f$. This will be accomplished by first factorizing $f \in \mathcal{O}(\tilde{X})$ through a continuous function F on Y and then proving that F is actually holomorphic. The existence of such a continuous factor F for f is equivalent to demonstrating that every $f \in \mathcal{O}(\tilde{X})$ is constant on the fibers of p . The validity of this geometric statement will be shown now using commutative algebra.

$\mathcal{O}(\tilde{X})$ is almost integral over $\mathcal{O}(X)$ (see § 1), and hence over the localization $S_x^{-1}A$ of A with respect to $S_x := \{g \in \mathcal{O}(X): g(x) \neq 0\}$ for every $x \in X$. Moreover,

$$S_x^{-1}A = \widetilde{S_x^{-1}\mathcal{O}(X)}$$

holds [3, V, 1.5, Corollary 1]. The localization $S_x^{-1}\mathcal{O}(X) = \mathcal{O}(X)_{m(x)}$ of the Stein algebra $\mathcal{O}(X)$ at the maximal ideal $m(x) := \{f \in \mathcal{O}(X): f(x) = 0\}$ is noetherian — even more, it's excellent [2, p. 35]. According to a theorem of Mori-Nagata, the integral closure of a noetherian integral domain is completely normal [7, 4.3, 3.6], implying

$$(*) \quad \mathcal{O}(\tilde{X}) \subset \bigcap_{x \in X} S_x^{-1} A.$$

For $f \in \mathcal{O}(\tilde{X})$, $a \in \tilde{X}$ and $b \in p^{-1}(p(a))$, it is now possible to conclude that $f(a) = f(b)$ is true. Let $x := \pi(a)$. Due to (*), functions $g \in S_x$ and $h \in A$ exist with $f = h/g \circ \pi$. Since a and b are equivalent with respect to the equivalence relation R , $f(a) = f(b)$ follows, and a continuous function $F: Y \rightarrow \mathbb{C}$ exists with $F \circ p = f$.

Since the Stein complex structure on Y is not in general the canonical ringed quotient structure, it is still necessary to verify that F is holomorphic in order to prove the density of A in $\mathcal{O}(\tilde{X})$. To that end, let $H \in \mathcal{O}(Y)$ and $G \in \mathcal{O}(Y)$ have the property that $H \circ p = h$ and $G \circ p = g \circ \pi$. Such functions exist because $p^*(\mathcal{O}(Y)) = \bar{A}$ holds. Then $F = H/G$ follows, and the germ $F_{p(a)}$ is the germ of a holomorphic function at $p(a)$, since the germ $G_{p(a)}$ of G at $p(a)$ is a unit. The surjectivity of p implies that F is holomorphic on Y , completing the proof of the theorem.

Note that the topology induced by $\mathcal{O}(\tilde{X})$ on any subalgebra A of $\mathcal{O}(\tilde{X})$ is the metrizable topology of uniform convergence on compact subsets of X . Because the closure $\bar{A}$ of A in $\mathcal{O}(\tilde{X})$ is its completion, $\bar{A}$ can be obtained without referring directly to $\mathcal{O}(\tilde{X})$. Thus the Main Theorem can be stated as follows:

If $\tilde{X}$ denotes the normalization of an irreducible Stein space X , then $\mathcal{O}(\tilde{X})$ is the completion of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$.

3. APPLICATIONS

In this section X will denote an irreducible Stein space with normalization $\pi: \tilde{X} \rightarrow X$, $\widetilde{\mathcal{O}(X)}$ will be the integral closure of the holomorphic functions $\mathcal{O}(X)$ on X , $\mathcal{O}(\tilde{X})$ the Fréchet algebra of weakly holomorphic functions on X (or equivalently, the Fréchet algebra of holomorphic functions $\mathcal{O}(\tilde{X})$ on $\tilde{X}$), and

$$S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\} \quad \text{for } x \in X.$$

Although the example given in the first section shows that the algebras $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ are not always equal, the inclusion (*) in the proof of the Main Theorem implies that they are locally equal in the following sense.

THEOREM 2. For every $x \in X$, the localizations of $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ with respect to S_x coincide.