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The group of all invertible  $n \times n$  upper triangular matrices will be denoted by  $B_n$ . Its subgroup consisting of all diagonal matrices is denoted by  $D_n$ . We have  $B_n = U_n \rtimes D_n$  where  $U_n$  is the closed connected subgroup of  $B_n$  consisting of all unipotent elements of  $B_n$ .

We start with some preliminary facts.

**THEOREM 1 (Lie-Kolchin).** *Every connected solvable affine algebraic group can be embedded in some  $B_n$  as a closed subgroup.*  $\square$

**COROLLARY.** *If  $G$  is a connected solvable affine group then  $G' \subset G_u$ .*  $\square$

**THEOREM 2 (Chevalley).** *If  $N$  is a closed normal subgroup of an affine group  $G$  then there exists a homomorphism  $f: G \rightarrow GL_n(k)$  such that  $\text{Ker } f = N$ .*  $\square$

For the proofs of Theorems 1 and 2 see, for instance, [5, Theorems 6.7 and 5.1.3].

**LEMMA 1.** *If  $f: G \rightarrow H$  is a surjective homomorphism of affine algebraic groups and  $N := \text{Ker } f$  then:*

- (i)  $f(G^0) = H^0$ ;
- (ii)  $f(G_u) = H_u$  and  $f(G_s) = H_s$ ;
- (iii)  $\dim G = \dim N + \dim H$ ;
- (iv) *If  $N$  and  $H$  are connected then  $G$  is connected.*

*Proof.* For the proofs of (i) and (iii) see for instance [4, Section 7.4]. (ii) follows from the fact that  $f$  preserves the Jordan decomposition [4, Theorem 2.4.8]. We shall sketch the proof of (iv). Since  $N$  is connected, we have  $N \subset G^0$ . By (i) we have  $f(G^0) = H^0 = H$ , and consequently  $G = NG^0 = G^0$ .  $\square$

We need a lemma to prove the centralizer theorem. For a more general version of this lemma see [2, Proposition (9.3)].

**LEMMA 2.** *Let  $N$  be a closed normal connected abelian unipotent subgroup of an affine group  $G$  and let  $s \in G_s$ . Then  $M := \{sus^{-1}u^{-1} : u \in N\}$  is a closed connected subgroup of  $N$ , the multiplication map  $\mu: M \times Z_N(s) \rightarrow N$  is bijective, and  $Z_N(s)$  is connected.*

*Proof.* Since  $N$  is abelian, the map  $f: N \rightarrow N$ , defined by  $f(u) = sus^{-1}u^{-1}$ , is a morphism of algebraic groups whose kernel is  $Z_N(s)$  and image  $M$ , so  $M$  is a closed connected subgroup of  $N$ . If  $x \in M \cap Z_N(s)$  then  $x = sus^{-1}u^{-1}$  for some  $u \in N$ . Since  $usu^{-1} = x^{-1}s = sx^{-1}$  is semi-simple and  $x$  is unipotent, the uniqueness of the Jordan decomposition implies that  $x = 1$ . Hence  $M \cap Z_N(s) = 1$  and so  $\mu$  is injective. By Lemma 1 (iii) we have  $\dim N = \dim M + \dim Z_N(s)$ , which implies that the homomorphism  $\mu$  is also surjective, i.e.,  $MZ_N(s) = N$ . The same argument shows that  $MZ_N(s)^0 = N$ , and so  $Z_N(s)$  must be connected.  $\square$

**THEOREM 3.** *If  $G$  is a connected solvable affine group and  $s \in G_s$  then  $Z_G(s)$  is connected and  $G = G_u Z_G(s)$ .*

*Proof.* We use induction on  $\dim G$ . If  $G$  is abelian the assertions are trivial. Otherwise let  $N$  be the last non-trivial term of the derived series of  $G$ . By the Corollary of Theorem 1,  $N$  is unipotent. We now apply Theorem 2 to this  $G$  and  $N$ . Let  $f$  be as in that theorem. We shall write  $\bar{x}$  for  $f(x)$  and  $\bar{G}$  for  $f(G)$ .

Let  $z \in G$  be such that  $\bar{z} \in Z_{\bar{G}}(\bar{s})$ . Then  $szs^{-1}z^{-1} \in N$ . By Lemma 2 there exists  $u \in N$  and  $v \in Z_N(s)$  such that  $szs^{-1}z^{-1} = sus^{-1}u^{-1} \cdot v$ . Since  $v$  commutes with  $u$  and  $s$ , and  $zsz^{-1} = v^{-1} \cdot usu^{-1}$ , it follows that  $v = 1$ . Thus  $u^{-1}z \in Z_G(s)$  and consequently we have a short exact sequence

$$1 \rightarrow Z_N(s) \hookrightarrow Z_G(s) \rightarrow Z_{\bar{G}}(\bar{s}) \rightarrow 1.$$

By Lemma 2,  $Z_N(s)$  is connected. By Lemma 1 (iii) we have  $\dim \bar{G} < \dim G$ . By induction hypothesis, we conclude that  $Z_{\bar{G}}(\bar{s})$  is connected and that  $\bar{G} = (\bar{G})_u \cdot Z_{\bar{G}}(\bar{s})$ . Now Lemma 1 (iv) implies that  $Z_G(s)$  is connected. By part (ii) of the same lemma we have  $f(G_u) = (\bar{G})_u$  and so  $f(G_u Z_G(s)) = (\bar{G})_u Z_{\bar{G}}(\bar{s}) = \bar{G}$ . Since  $N \subset G_u$ , it follows that  $G = G_u Z_G(s)$ .  $\square$

We now proceed to prove the main results about the structure of connected solvable affine groups. But first we need two lemmas.

**LEMMA 3.** *Let  $S \subset B_n$  be a commuting set of semisimple elements. Then there exists  $b \in B_n$  such that  $b^{-1}Sb \subset D_n$ .*

*Proof.* It is an elementary fact of linear algebra that there exists  $a \in GL_n(k)$  such that  $a^{-1}Sa \subset D_n$ . Hence if  $M_n(k)$  is the algebra of  $n$  by  $n$  matrices over  $k$  and  $A$  its subalgebra generated by  $S$ , we know that  $A$  is semisimple (and commutative). Let  $V := k^n$  be the space of column

vectors and let  $e_1, \dots, e_n$  be its standard basis. We shall view  $V$  as a left  $M_n(k)$ -module via matrix multiplication. The subspace  $V_i$  spanned by the vectors  $e_1, \dots, e_i$  is an  $A$ -submodule of  $V$  for each  $i$ . Since  $A$  is semisimple, there exist  $v_i \in V_i \setminus V_{i-1}$ ,  $1 \leq i \leq n$ , such that  $Av_i = kv_i$ . Thus if  $b$  is the matrix whose  $i$ -th column is  $v_i$ ,  $1 \leq i \leq n$ , then  $b \in B_n$  and  $b^{-1}Sb \subset D_n$ .  $\square$

LEMMA 4. *If  $G$  is a connected solvable affine group,  $T \subset G_s$  a closed subgroup, and  $G = G_u T$  then  $T$  is a torus and  $G = G_u \rtimes T$ .*

*Proof.* By the Lie-Kolchin theorem we may assume that  $G$  is a closed subgroup of some  $B_n$ . By using the projection map  $B_n \rightarrow D_n$  we obtain a short exact sequence  $1 \rightarrow G_u \hookrightarrow G \xrightarrow{p} D \rightarrow 1$ , where  $D \subset D_n$  is a torus. Since  $D = p(G) = p(G_u T) = p(T)$ , Lemma 1 (i) implies that  $p(T^0) = D$ . Thus  $G = G_u T^0$  and using  $T \cap G_u = 1$  we conclude that  $T = T^0$ . In particular  $T$  is abelian and by Lemma 3 we may assume that  $T \subset D_n$ , i.e.,  $T = D$ . Since  $B_n = U_n \rtimes D_n$ ,  $G_u \subset U_n$ ,  $T = D \subset D_n$ , and  $G = G_u T$ , it follows that  $G = G_u \rtimes T$ .  $\square$

THEOREM 4. *Let  $G$  be a connected solvable affine group. Then  $G = G_u \rtimes T$  where  $T$  is a maximal torus. In particular,  $G_u$  is connected.*

*Proof.* We use induction on  $\dim G$ . Assume first that  $G_s \subset Z(G)$ . Then  $G_s = Z(G)_s$  is a closed subgroup of  $G$  and  $G = G_u G_s$ . The assertion then follows from Lemma 4. Now assume that there exists  $s \in G_s \setminus Z(G)$ . Then  $Z_G(s)$  is a proper closed subgroup of  $G$ , see e.g. [4, Section 8.2]. By Theorem 3 it is connected and  $G = G_u Z_G(s)$ . By induction hypothesis there exists a torus  $T$  such that  $Z_G(s) = Z_G(s)_u T$ . Then  $G = G_u Z_G(s) = G_u T$  and  $G = G_u \rtimes T$  by Lemma 4.  $\square$

THEOREM 5. *Let  $G = G_u \rtimes T$  be a connected solvable affine group. Then every  $s \in G_s$  is conjugate to an element of  $T$ .*

*Proof.* We use induction on  $\dim G$ . We have  $s = ut$  where  $u \in G_u$  and  $t \in T$ . If  $G$  is abelian then  $u = 1$  and  $s = t$ . Otherwise let  $N$  be the last non-trivial term of the derived series of  $G$ . By the corollary of Theorem 1 we have  $N \subset G_u$ . Hence  $N$  is a closed connected normal abelian unipotent subgroup of  $G$ . By Theorem 2 and the induction hypothesis there exists  $x \in G$  such that  $xsx^{-1} = tv$  where  $v \in N$ . By Lemma 2,  $v = t^{-1}utu^{-1}z$  where  $u \in N$  and  $z \in Z_N(t)$ . Hence  $xsx^{-1} = utu^{-1}z$ . Since  $xsx^{-1}, utu^{-1} \in G_s$ ,  $z \in G_u$ , and  $z$  commutes with  $u$  and  $t$ , it follows that  $z = 1$  and consequently  $xsx^{-1} = utu^{-1}$ .  $\square$